Effect of Coulomb Interaction and Disorder on Density of States in Conventional Superconductors

Takanobu Jujo *

Division of Materials Science, Graduate School of Science and Technology, Nara Institute of Science and Technology, Ikoma, Nara 630-0101, Japan

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The density of states of the disordered s-wave superconductor is calculated perturbatively. The effect of Coulomb interaction on diffusively moving electrons in the normal state has been known before, but in the superconducting state both diffuson and the screened Coulomb interaction are modified. Therefore, the correction to the density of states in the superconducting state exhibits an energy dependence different from that of the normal state. There is a dip structure in the correction part because the interaction has a peak at twice the energy of the superconducting gap. The Coulomb interaction and the superconducting fluctuation cannot be treated separately because the density fluctuation is coupled to the phase fluctuation in the superconducting state. This coupling results in the absence of divergence around the gap edge in the correction part, which suggests the validity of this perturbation calculation.

1. Introduction

The conventional s-wave superconductor is not affected by the impurity scattering itself because nonmagnetic impurities do not break the symmetry of s-wave superconductors. [1] In general there exist interactions between electrons in superconductors, and the Coulomb interaction changes low-energy properties of electrons moving diffusively by disorder.[2–4] This is how the scattering by nonmagnetic impurities reduces the transition temperature of s-wave superconductors. [5–7] Thus, the correlation between interactions and disorder in superconductors has been an interesting research subject.

Studies on correlation between the Coulomb interaction and impurity scattering have been mainly conducted in the normal state, and physical quantities such as specific heat and conductivity have been calculated not only in the three-dimensional case,[3,4] but also in the two-dimensional system.[8–10] The deviation of physical properties from

*E-mail address: jujo@ms.aist-nara.ac.jp

those of a Fermi liquid is caused by the suppression of low-energy electronic states owing to the Coulomb interaction enhanced by diffuson. This suppression of the density of states (DOS) near the Fermi level is known as the Altshuler-Aronov effect. Not only the screened Coulomb interaction but the superconducting fluctuation is also enhanced by the diffusive motion of electrons, and this effect results in the suppression of the DOS above the superconducting transition temperature.[11,12]

There have been several measurements on the DOS both in the ultrathin film[13,14] (whose thickness is comparable to the coherence length) and the three-dimensional system.[15–17] These studies mainly focus on the physical properties near the superconductor-insulator transition, especially the variation of the size of the superconducting gap and its spatial distribution when the disorder is increased. For this reason, although the DOS exhibits an energy dependence similar to that of the Altshuler-Aronov effect both above and below the superconducting transition temperature, this energy dependence is treated as a uniform background. Therefore, the dependence of the DOS on energy in the superconducting state has not been precisely investigated.

In this study, we calculated the correction to the DOS in the superconducting state of a three-dimensional system. We considered the weakly localized regime in which the expansion parameter of the perturbation is $1/k_F l$ (k_F and l being the Fermi wave number and the mean free path, respectively). We also assume the dirty limit ($\Delta\tau \ll 1$. Δ and τ being the superconducting gap and the relaxation time, respectively). In the calculation the Coulomb interaction is included consistently with the superconducting correlation.

Although the Altshuler-Aronov effect in the superconducting state has been studied with use of the Coulomb interaction and diffuson of the normal state,[18,19] the Coulomb interaction and the effect of disorder are modified in the superconducting state. The density fluctuation couples to the fluctuation of the phase of the superconducting order parameter.[20] In addition, because there is an energy gap in the superconducting state, the diffusive motion of quasiparticles is modified and the calculation in the normal state does not hold at low energy. Therefore, in the vicinity of the energy gap, the correction to the DOS also differs from that of the normal state.

This paper is organized as follows. In Sect. 2, the expression for DOS is derived, after discussing the model and the approximations required to calculate the correction to the DOS. In Sect. 3, after discussing the temperature dependence and diffuson in the superconducting state, the results of numerical calculations at absolute zero are presented. In Sect. 4, a short summary is provied along with a discussion of the effects that are not included in this paper.

2. Formulation

The Hamiltonian is given by

$$\mathcal{H} = \sum_{k,\sigma} \xi_k c^\dagger_{k,\sigma} c_{k,\sigma} + \sum_q \omega_q b^\dagger b_q + \frac{g_{ph}}{\sqrt{N^3}} \sum_{k,q,\sigma} (b_q + b^\dagger_{-q}) c^\dagger_{k+q,\sigma} c_{k,\sigma} + \frac{1}{\sqrt{N^3}} \sum_{k,k',\sigma} u_{k-k'} c^\dagger_{k,\sigma} c_{k',\sigma} + \frac{1}{2N^3} \sum_{k,k',q,\sigma,\sigma'} v_q c^\dagger_{k,\sigma} c_{k+q,\sigma} c^\dagger_{k',\sigma'} c_{k'-q,\sigma'}. \tag{1}$$

ξ_k and ω_q are the dispersions of electrons and phonons, respectively. The third and fourth terms represent the interaction between electrons and phonons and the effect of impurity scattering, respectively. We assume that ω_q does not depend on q and that it takes a constant value $\omega_q = \omega_E$. The fifth term represents the Coulomb interaction between electrons and $v_q = 4\pi e^2/q^2$. N^3 is the number of sites. We consider the three-dimensional system, and k and q are wave number vectors in this space. We set $\hbar = 1$ in this paper.

The correction to the DOS is given by

$$\rho'(\epsilon) = \frac{-1}{\pi} \mathrm{Im} \frac{1}{N^3} \sum_{\boldsymbol{k}} \mathrm{Tr}[\hat{G}_k \hat{G}'_k \hat{G}_k]_{i\epsilon_n \to \epsilon + i0^+}. \tag{2}$$

Hereafter, we use the notation $k = (\boldsymbol{k}, \epsilon_n)$, where $\boldsymbol{k}$ is a wave number vector in the three dimensional space and $\epsilon_n = \pi T(2n-1)$ is the Matsubara frequency with T the temperature. The term Im indicates the imaginary part, and $i\epsilon_n \to \epsilon + i0^+$ means the analytic continuation, with 0^+ an infinitesimal positive quantity ($i = \sqrt{-1}$). $\hat{G}_k$ is the Green function of electrons and includes the effects of the impurity scattering and the electron-phonon interaction with Born and mean-field approximations,[21] respectively,

$$\hat{G}_k = \frac{1}{(i\tilde{\epsilon}_n)^2 - \xi_{\boldsymbol{k}}^2 - \tilde{\Delta}^2} \begin{pmatrix} i\tilde{\epsilon}_n + \xi_{\boldsymbol{k}} & \tilde{\Delta} \\ \tilde{\Delta} & i\tilde{\epsilon}_n - \xi_{\boldsymbol{k}} \end{pmatrix}. \tag{3}$$

Here, $\tilde{\epsilon}_n$ and $\tilde{\Delta}$ are determined by the following equation:

$$(i\epsilon_n - i\tilde{\epsilon}_n)\hat{\tau}_3 + (\tilde{\Delta} - \Delta)\hat{\tau}_1 = \frac{n_i u^2}{N^3} \sum_{\boldsymbol{k}} \hat{\tau}_3 \hat{G}_k \hat{\tau}_3 \tag{4}$$

where $\hat{\tau}_3 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ and $\hat{\tau}_1 = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. ($n_i$ and u represent the concentration of impurities and the magnitude of the impurity potential, respectively.) Δ is the superconducting

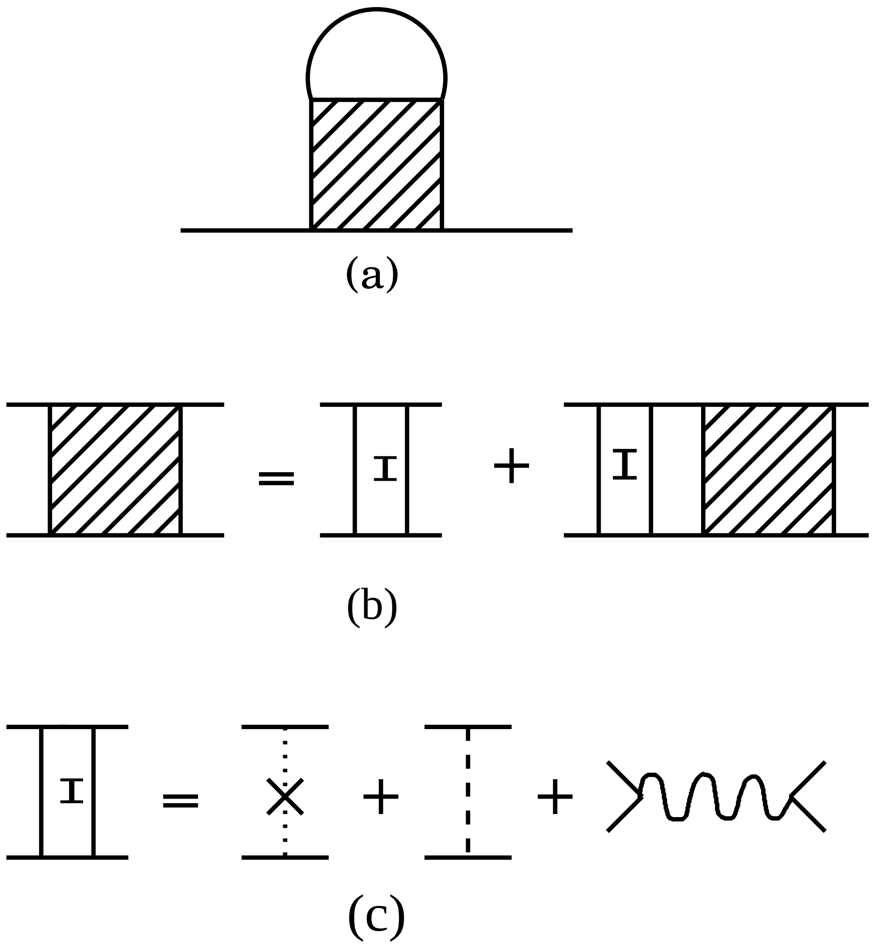

Fig. 1. (a) The diagrammatic representation of the correction to the DOS. The solid line indicates the propagator of electrons $\hat{G}_k$, and the shaded square includes the effects of interactions. (b) The interaction effect is obtained by solving this equation. The square with "I" included indicates the irreducible part. (c) The irreducible part. The dotted line with a cross represents the scattering by impurities. The dashed line means the electron-phonon interaction. The wavy line represents the Coulomb interaction.

gap determined by the gap equation,

$$\Delta\hat{\tau}_1 = \frac{g_{ph}^2}{\omega_E}\frac{2T}{N^3}\sum_k \hat{\tau}_3\hat{G}_k\hat{\tau}_3. \tag{5}$$

The effects of interactions beyond the mean-field approximation are included in $\hat{G}'_k$, and its diagrammatic representation is shown in Fig. 1. The three interaction terms in Fig. 1(c) combined with the equation represented by Fig. 1(b) give the physical effects that are predominant at low energy. The first term (the scattering by impurities) induces the diffusive motion of electrons, the second term (the interaction of electrons with phonons) results in the superconducting fluctuation, and the third term gives the screened Coulomb interaction.

We obtain $\hat{G}'_k$ as follows. The components of $\hat{G}'_k$ are given by

$$(\hat{G}'_k)_{jj'} = \frac{-2T}{N^3} \sum_q \tilde{\gamma}^{ij,i'j'}_{k,k-q} (\hat{G}_{k-q})_{ii'} \tag{6}$$

in which i, j, i', j' are indices specifying rows and columns of 2×2 matrices; hereafter the summation is taken over repeated indices. $\tilde{\gamma}^{ij,i'j'}_{k,k-q}$ is given by

$$\tilde{\gamma}^{ij,i'j'}_{k,k-q} = \left[\delta_{i,s}\delta_{j,t} + n_i u^2 M^{ij,lm}_{\epsilon_n,\epsilon_n-\omega_l} (\hat{\tau}_3)_{sl} (\hat{\tau}_3)_{mt}\right] \gamma^{st,s't'}_q \left[\delta_{i',s'}\delta_{j',t'} + (\hat{\tau}_3)_{l's'} (\hat{\tau}_3)_{t'm'} n_i u^2 M^{l'm',i'j'}_{\epsilon_n,\epsilon_n-\omega_l}\right], \tag{7}$$

where $\delta_{i,s}$ is Kronecker's delta function. $\gamma^{ij,i'j'}_q$ and $M^{ij,i'j'}_{\epsilon_n,\epsilon_n-\omega_l}$ are given by the following equations.

$$\begin{aligned} \gamma^{ij,i'j'}_q = & \left[\frac{g^2_{ph}}{\omega_E} (\hat{\tau}_3)_{i'i} (\hat{\tau}_3)_{jj'} + \frac{v_q}{2} (\hat{\tau}_3)_{ij} (\hat{\tau}_3)_{i'j'}\right] \\ & + \left[\frac{g^2_{ph}}{\omega_E} (\hat{\tau}_3)_{li} (\hat{\tau}_3)_{jm} + \frac{v_q}{2} (\hat{\tau}_3)_{ij} (\hat{\tau}_3)_{lm}\right] 2T \sum_{\epsilon_n} M^{lm,l'm'}_{\epsilon_n,\epsilon_n-\omega_l} \gamma^{l'm',i'j'}_q \end{aligned} \tag{8}$$

and

$$M^{ij,i'j'}_{\epsilon_n,\epsilon_n-\omega_l} = \frac{1}{N^3} \sum_{\boldsymbol{k}} (\hat{G}_k)_{jj'} (\hat{G}_{k-q})_{i'i} + \frac{n_i u^2}{N^3} \sum_{\boldsymbol{k}} (\hat{G}_k)_{jm} (\hat{G}_{k-q})_{li} (\hat{\tau}_3)_{l'l} (\hat{\tau}_3)_{mm'} M^{l'm',i'j'}_{\epsilon_n,\epsilon_n-\omega_l}. \tag{9}$$

These equations are solved by introducing 4×4 matrices such as

$$\hat{M} := \begin{pmatrix} M^{11,11} & M^{11,22} & M^{11,12} & M^{11,21} \\ M^{22,11} & M^{22,22} & M^{22,12} & M^{22,21} \\ M^{12,11} & M^{12,22} & M^{12,12} & M^{12,21} \\ M^{21,11} & M^{21,22} & M^{21,12} & M^{21,21} \end{pmatrix}. \tag{10}$$

Then, for example,

$$(\hat{\tau}_3)_{i'i} (\hat{\tau}_3)_{jj'} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{11}$$

and

$$(\hat{\tau}_3)_{ij} (\hat{\tau}_3)_{i'j'} = \begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{12}$$

By solving Eq. (9), the 4×4 matrix corresponding to $2T\sum_{\epsilon_n} M^{ij,i'j'}_{\epsilon_n,\epsilon_n-\omega_l}$ is written as follows:

$$\frac{\pi\rho_0}{2}\begin{pmatrix} (\chi_3+\chi_0)\hat{\tau}_0/2-(\chi_3-\chi_0)\hat{\tau}_1/2 & \chi'(\hat{\tau}_0-\hat{\tau}_1) \\ \chi'(\hat{\tau}_0-\hat{\tau}_1) & (\chi_2+\chi_1)\hat{\tau}_0/2-(\chi_2-\chi_1)\hat{\tau}_1/2 \end{pmatrix}. \tag{13}$$

Here, $\hat{\tau}_0 = \left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$, and $\rho_0 = mk_F/\pi^2$ is the noninteracting density of states at the Fermi level.

$$\chi_i = 2T\sum_{\epsilon_n}\frac{X_{\epsilon_n+\omega_l,\epsilon_n}/\alpha}{1-2X_{\epsilon_n+\omega_l,\epsilon_n}}(h_i+g_{\epsilon_n+\omega_l}g_{\epsilon_n}+h'_i f_{\epsilon_n+\omega_l}f_{\epsilon_n})-\frac{2}{\pi}(\delta_{i,3}+\delta_{i,0}) \tag{14}$$

(the second term is necessary when the integration over $\xi_{\boldsymbol{k}}$ is performed before the summation over ϵ_n[22]), and

$$\chi' = T\sum_{\epsilon_n}\frac{X_{\epsilon_n+\omega_l,\epsilon_n}/\alpha}{1-2X_{\epsilon_n+\omega_l,\epsilon_n}}(g_{\epsilon_n+\omega_l}f_{\epsilon_n}-f_{\epsilon_n+\omega_l}g_{\epsilon_n}). \tag{15}$$

$g_{\epsilon_n} = -i\epsilon_n/\zeta_{\epsilon_n}$, $f_{\epsilon_n} = -\Delta/\zeta_{\epsilon_n}$, $\zeta_{\epsilon_n} = \sqrt{\epsilon_n^2+\Delta^2}$, $\alpha := n_i u^2 mk_F/2\pi$,

$$h_3 = h_0 = h'_0 = h'_1 = 1, \tag{16}$$

and

$$h'_3 = h_2 = h'_2 = h_1 = -1. \tag{17}$$

α is related to the relaxation time by the impurity scattering: $\tau = 1/2\alpha = 1/\pi\rho_0 n_i u^2$.

$$X_{\epsilon_n,\epsilon_{n'}} := \int_{FS}\frac{2\alpha+\zeta_{\epsilon_n}+\zeta_{\epsilon_{n'}}}{(2\alpha+\zeta_{\epsilon_n}+\zeta_{\epsilon_{n'}})^2+(v_k\cdot q)^2} = \frac{2\alpha}{v_F q}\arctan\left(\frac{v_F q}{2\alpha+\zeta_{\epsilon_n}+\zeta_{\epsilon_{n'}}}\right). \tag{18}$$

($\int_{FS}$ indicates the integration over the Fermi surface.) In the case of a dirty limit ($v_F q/2\alpha \ll 1$, $(\zeta_{\epsilon_n}+\zeta_{\epsilon_{n'}})/2\alpha \ll 1$)

$$X_{\epsilon_n,\epsilon_{n'}} \simeq \frac{2\alpha-(D_\alpha q^2+\zeta_{\epsilon_n}+\zeta_{\epsilon_{n'}})}{4\alpha} \tag{19}$$

with the diffusion constant $D_\alpha = v_F^2\tau/3$ (v_F is the Fermi velocity).

The indices i of χ_i correspond to those of Pauli matrices ($\hat{\tau}_i$). Using Eq. (13),

$$\left(\frac{\pi\rho_0}{2}\right)^{-1}(2T\sum_{\epsilon_n}M^{ij,i'j'}_{\epsilon_n,\epsilon_n-\omega_l})(\hat{\tau}_{0,1})_{ii'} = \chi_{0,1}(\hat{\tau}_{0,1})_{jj'}, \tag{20}$$

and

$$\left(\frac{\pi\rho_0}{2}\right)^{-1}(2T\sum_{\epsilon_n}M^{ij,i'j'}_{\epsilon_n,\epsilon_n-\omega_l})(\hat{\tau}_3+i\hat{\tau}_2)_{ii'} = (\chi_3+2\chi')(\hat{\tau}_3)_{jj'}+(\chi_2+2\chi')(i\hat{\tau}_2)_{jj'}. \tag{21}$$

$\hat{\tau}_2 = \left(\begin{smallmatrix}0&-i\\i&0\end{smallmatrix}\right)$. These equations indicate that the density fluctuation ($\hat{\tau}_3$) couples to the phase fluctuation ($\hat{\tau}_2$) in the presence of a finite value of the superconducting gap (the mixing term χ' vanishes when $\Delta = 0$), and the amplitude fluctuation ($\hat{\tau}_1$) decouples from other modes in the presence of a particle-hole symmetry.

Then, the solution for Eq. (8) is written in the 4×4 matrix form as follows:

$$\hat{\gamma}_q = \left(\frac{\pi\rho_0}{2}\right)^{-1} \begin{pmatrix} \Gamma_3(q)(\hat{\tau}_0 - \hat{\tau}_1) + \Gamma_0(q)(\hat{\tau}_0 + \hat{\tau}_1) & \Gamma'(q)(\hat{\tau}_0 - \hat{\tau}_1) \\ \Gamma'(q)(\hat{\tau}_0 - \hat{\tau}_1) & \Gamma_2(q)(\hat{\tau}_0 - \hat{\tau}_1) + \Gamma_1(q)(\hat{\tau}_0 + \hat{\tau}_1) \end{pmatrix}. \tag{22}$$

Here,

$$\Gamma_3(q) = \frac{(p + c_q)(1/p + \chi_2)/2}{(1/p + \chi_2)[1 - (p + c_q)\chi_3] + 4(p + c_q)(\chi')^2}, \tag{23}$$

$$\Gamma_2(q) = \frac{-[1 - (p + c_q)\chi_3]/2}{(1/p + \chi_2)[1 - (p + c_q)\chi_3] + 4(p + c_q)(\chi')^2}, \tag{24}$$

$$\Gamma'(q) = \frac{(p + c_q)\chi'}{(1/p + \chi_2)[1 - (p + c_q)\chi_3] + 4(p + c_q)(\chi')^2}, \tag{25}$$

$$\Gamma_0(q) = \frac{p/2}{1 - p\chi_0}, \tag{26}$$

and

$$\Gamma_1(q) = \frac{-1/2}{1/p + \chi_1}. \tag{27}$$

Here, $p := mk_F g_{ph}^2/2\pi\omega_E$ indicates the coupling constant between electrons and phonons and $c_q := mk_F v_q/2\pi$.

Using the above results, the correction to the DOS is written as follows.

$$\begin{aligned} \frac{-1}{\pi N^3}\sum_k \mathrm{Tr}[\hat{G}_k \hat{G}'_k \hat{G}_k] &\simeq \rho_0 \frac{3\sqrt{3\tau}}{2\pi(k_F l)^2} 2T \sum_{\omega_l} \int dx \sqrt{x} \\ &\times \frac{\Gamma_i(q)(h_i + g_{\epsilon_n} g_{\epsilon_n - \omega_l} + h'_i f_{\epsilon_n} f_{\epsilon_n - \omega_l}) + 2\Gamma'(q)(f_{\epsilon_n} g_{\epsilon_n - \omega_l} - g_{\epsilon_n} f_{\epsilon_n - \omega_l})}{(x + \zeta_{\epsilon_n} + \zeta_{\epsilon_n - \omega_l})^2} g_{\epsilon_n}. \end{aligned} \tag{28}$$

($x = D_\alpha q^2$.) Here we use the approximate expression Eq. (19), and introduce the upper limits of $|\omega_l|$ and $D_\alpha q^2$ (which are on the order of 2α and will be specified when the numerical calculation is performed in Sect. 3). (The high energy parts from $|\omega_l|/2\alpha \gg 1$ or $v_F q/2\alpha \gg 1$ are assumed to be included in the parameters of the electronic states. In fact, $1/(1 - 2X_{\epsilon_n, \epsilon_n - \omega_l}) \simeq 1$ in this range, and the correction term is reduced to the usual Fock term because the diffuson propagator is absent.)

2.1 Normal state

In this subsection, we show that the expressions previously studied in the normal state [3,4,11,12] are obtained by setting $\Delta = 0$ in the above expressions. For $\Delta = 0$ and after analytic continuation ($i\omega_l \to \omega + i0^+$) χ_i ($i = 0, 1, 2, 3$) and χ' are written as follows.

$$\chi_3 = \chi_0 = \frac{2}{\pi}\frac{-D_\alpha q^2}{D_\alpha q^2 - i\omega}, \tag{29}$$

$$\frac{1}{p}+\chi_2 = \frac{1}{p}+\chi_1 = \frac{2}{\pi}\int d\epsilon \left[\frac{\tanh(\epsilon/2T_c)}{2\epsilon} + \frac{-\tanh(\epsilon/2T)}{2\epsilon+\omega+iD_\alpha q^2}\right] \simeq \frac{2}{\pi}\left[\ln\left(\frac{T}{T_c}\right) + \frac{\pi}{8T}(D_\alpha q^2 - i\omega)\right] \tag{30}$$

(T_c is the superconducting transition temperature) and $\chi' = 0$. Then, $\Gamma_i(q)$ and $\Gamma'(q)$ are given by

$$\Gamma_3(q) = \frac{(p+c_q)(1/p+\chi_2)/2}{(1/p+\chi_2)[1-(p+c_q)\chi_3]} \simeq \frac{-1/2}{\chi_3}, \tag{31}$$

$$\Gamma_2(q) = \frac{-1/2}{1/p+\chi_2} = \Gamma_1(q), \tag{32}$$

$$\Gamma_0(q) = \frac{p/2}{1-p\chi_3}, \tag{33}$$

and $\Gamma'(q) = 0$.

The correction to the DOS in the normal state is given by the following equation:

$$\rho'(\epsilon) = \rho'_{sf}(\epsilon) + \rho'_{cl}(\epsilon) \tag{34}$$

with

$$\rho'_{sf}(\epsilon) \simeq \rho_0 \frac{12\sqrt{3}\tau}{(2\pi k_F l)^2}\int d\omega \int dx\sqrt{x}\,\mathrm{Im}\left\{\frac{2i\coth(\frac{\omega}{2T})\mathrm{Im}[\Gamma_2(q)] + \tanh(\frac{\epsilon-\omega}{2T})\Gamma_2(q)}{[x-i(2\epsilon-\omega)]^2}\right\} \tag{35}$$

and

$$\rho'_{cl}(\epsilon) \simeq \rho_0 \frac{6\sqrt{3}\tau}{(2\pi k_F l)^2}\int d\omega \int dx\sqrt{x}\,\mathrm{Im}\left\{\frac{\tanh(\frac{\epsilon-\omega}{2T})[\Gamma_3(q)+\Gamma_0(q)]}{(x-i\omega)^2}\right\}. \tag{36}$$

$\rho'_{sf}(\epsilon)$ and $\rho'_{cl}(\epsilon)$ include the effects of the superconducting fluctuation above T_c[11,12] and the screened Coulomb interaction enhanced by diffuson,[3,4] respectively.

3. Results

3.1 The temperature dependence of the correction to the density of states

In this subsection, we show that the temperature dependence of the correction to DOS is small at low temperature $T \ll \Delta$.

After analytic continuation, Eq. (14) is written as follows.

$$\chi_i = \int \frac{d\epsilon}{2\pi i}\left[\tanh\left(\frac{\epsilon}{2T}\right)(\kappa^i_{++} - \kappa^i_{+-}) + \tanh\left(\frac{\epsilon+\omega}{2T}\right)(\kappa^i_{+-} - \kappa^i_{--})\right] - \frac{2}{\pi}(\delta_{i,3}+\delta_{i,0}) \tag{37}$$

with

$$\kappa^i_{ss'} = \frac{X^{ss'}_{\epsilon+\omega,\epsilon}/\alpha}{1-2X^{ss'}_{\epsilon+\omega,\epsilon}}(h_i + g^s_{\epsilon+\omega}g^{s'}_{\epsilon} + h'_i f^s_{\epsilon+\omega} f^{s'}_{\epsilon}). \tag{38}$$

χ' is obtained by replacing $\kappa^i_{ss'}$ in Eq. (37) with $i \neq 3, 0$ by

$$\kappa'_{ss'} = \frac{X^{ss'}_{\epsilon+\omega,\epsilon}/\alpha}{1-2X^{ss'}_{\epsilon+\omega,\epsilon}}(g^s_{\epsilon+\omega}f^{s'}_{\epsilon} - f^s_{\epsilon+\omega}g^{s'}_{\epsilon})/2. \tag{39}$$

Here, $s, s' = +$ (retarded) or $-$ (advanced), $g^s_\epsilon = -\epsilon/\zeta^s_\epsilon$, and $f^s_\epsilon = -\Delta/\zeta^s_\epsilon$ with $\zeta^\pm_\epsilon = \sqrt{\Delta^2-\epsilon^2}\theta(\Delta - |\epsilon|) - i\mathrm{sgn}(\pm\epsilon)\sqrt{\epsilon^2-\Delta^2}\theta(|\epsilon|-\Delta)$ [$\theta(\cdot)$ is a step function].

$$X^{ss'}_{\epsilon,\epsilon'} = \int_{FS} \frac{2\alpha + \zeta^s_\epsilon + \zeta^{s'}_{\epsilon'}}{(2\alpha + \zeta^s_\epsilon + \zeta^{s'}_{\epsilon'})^2 + (v_k\cdot q)^2} \simeq \frac{2\alpha - (D_\alpha q^2 + \zeta^s_\epsilon + \zeta^{s'}_{\epsilon'})}{4\alpha}. \tag{40}$$

From Eq. (37),

$$\mathrm{Im}\chi_i = \int \frac{d\epsilon}{2\pi}\left[\tanh\left(\frac{\epsilon+\omega}{2T}\right) - \tanh\left(\frac{\epsilon}{2T}\right)\right]\mathrm{Re}(\kappa^i_{++} - \kappa^i_{+-}). \tag{41}$$

$\mathrm{Re}(\kappa^i_{++} - \kappa^i_{+-})$ takes finite values only for $|\epsilon+\omega| > \Delta$ and $|\epsilon| > \Delta$. Then, $\mathrm{Im}\chi_i$ is exponentially small for $|\omega| < 2\Delta$ except for $T \simeq T_C$, and is negligible in this region.

We consider the correction to the DOS for $|\epsilon| < \Delta$ and $|\epsilon| > \Delta$ separately in the following. First, we consider the case of $|\epsilon| < \Delta$. After performing the analytic continuation of Eq. (28), the imaginary part is written as follows.

$$\begin{aligned}\rho'(\epsilon) \simeq\ & \frac{\rho_0\epsilon}{\sqrt{\Delta^2-\epsilon^2}}\frac{-6\sqrt{3\tau}}{(2\pi k_F l)^2}\int d\omega \int dx\sqrt{x}\left[\coth\left(\frac{\omega}{2T}\right) + \tanh\left(\frac{\epsilon-\omega}{2T}\right)\right] \\ & \times \mathrm{Im}\left\{\frac{\mathrm{Im}[\Gamma_i(q)](h_i + g_\epsilon g^+_{\epsilon-\omega} + h'_i f_\epsilon f^+_{\epsilon-\omega}) + 2\mathrm{Im}[\Gamma'(q)](f_\epsilon g^+_{\epsilon-\omega} - g_\epsilon f^+_{\epsilon-\omega})}{(x + \zeta_\epsilon + \zeta^+_{\epsilon-\omega})^2}\right\}\end{aligned} \tag{42}$$

($\zeta_\epsilon = \sqrt{\Delta^2-\epsilon^2}$, $g_\epsilon = -\epsilon/\zeta_\epsilon$, and $f_\epsilon = -\Delta/\zeta_\epsilon$). The imaginary part is finite ($\mathrm{Im}\{\cdot\} \neq 0$) only for $|\epsilon - \omega| > \Delta$. For $|\omega| < 2\Delta$, $\mathrm{Im}\Gamma_i$ and $\mathrm{Im}\Gamma'$ are exponentially small at low temperature, as noted above. The factor $\coth(\omega/2T) + \tanh[(\epsilon-\omega)/2T]$ is also exponentially small for $|\epsilon| < \Delta$ and $|\omega| > 2\Delta$. Then, the correction to the DOS is negligible for $|\epsilon| < \Delta$ except for $T \simeq T_C$.

On the other hand, for $|\epsilon| > \Delta$, the imaginary part of Eq. (28) after the analytic continuation is written as follows:

$$\rho'(\epsilon) \simeq \frac{\rho_0|\epsilon|}{\sqrt{\epsilon^2-\Delta^2}} \frac{-3\sqrt{3\tau}}{(2\pi k_F l)^2} \int d\omega \int dx \sqrt{x}$$
$$\times \, \mathrm{Im}\Big\{ 2\mathrm{coth}\left(\frac{\omega}{2T}\right) \frac{\mathrm{Im}[\Gamma_i(q)](h_i + g_\epsilon^+ g_{\epsilon-\omega}^+ + h_i' f_\epsilon^+ f_{\epsilon-\omega}^+) + 2\mathrm{Im}[\Gamma'(q)](f_\epsilon^+ g_{\epsilon-\omega}^+ - g_\epsilon^+ f_{\epsilon-\omega}^+)}{(x+\zeta_\epsilon^+ + \zeta_{\epsilon-\omega}^+)^2}$$
$$+ \tanh\left(\frac{\epsilon-\omega}{2T}\right) \sum_{s=\pm} s \frac{\Gamma_i(q)(h_i + g_\epsilon^+ g_{\epsilon-\omega}^s + h_i' f_\epsilon^+ f_{\epsilon-\omega}^s) + 2\Gamma'(q)(f_\epsilon^+ g_{\epsilon-\omega}^s - g_\epsilon^+ f_{\epsilon-\omega}^s)}{(x+\zeta_\epsilon^+ + \zeta_{\epsilon-\omega}^s)^2} \Big\}. \tag{43}$$

In this equation the coefficient of $\coth(\omega/2T)$ is exponentially small for $|\omega| < 2\Delta$ owing to the existence of $\mathrm{Im}\Gamma_i$ and $\mathrm{Im}\Gamma'$, and the coefficient of $\tanh[(\epsilon-\omega)/2T]$ vanishes for $|\epsilon-\omega| < \Delta$ (the imaginary part is absent). This indicates that the dependence of $\rho'(\epsilon)$ for $|\epsilon| > \Delta$ on temperature is weak for $T \ll \Delta$. This small dependence of $\rho'(\epsilon)$ on temperature is consistent with exponentially small values of $\rho'(\epsilon)$ for $|\epsilon| < \Delta$ at low temperature. Thus, we perform the numerical calculations at $T = 0$ and $\epsilon > \Delta$ in Sect. 3.3.

3.2 Diffuson in the superconducting state

The diffuson propagator is usually represented by $1/(D_\alpha q^2 - i\omega)$. However, in the superconducting state [Eq. (43), $x = D_\alpha q^2$] it is given by $1/(x + \zeta_\epsilon^+ + \zeta_{\epsilon-\omega}^\pm) = 1/\{x - i[\mathrm{sgn}(\epsilon)\sqrt{\epsilon^2-\Delta^2} \pm \mathrm{sgn}(\epsilon-\omega)\sqrt{(\epsilon-\omega)^2-\Delta^2}]\}$ for $|\epsilon|, |\epsilon-\omega| > \Delta$ (the diffusive motion of quasiparticles is effective above the superconducting gap). Another singularity exists at $\omega = 2\epsilon$ in the case of $1/(x + \zeta_\epsilon^+ + \zeta_{\epsilon-\omega}^+)$ in addition to the pole at $\omega = 0$ in $1/(x + \zeta_\epsilon^+ + \zeta_{\epsilon-\omega}^-)$. In this subsection, we illustrate that the divergence by this additional pole is absent when the particle-number conservation is preserved in the integration of Eq. (43).

By performing the analytic calculation,

$$\chi_3(\boldsymbol{q}=\boldsymbol{0}) = \frac{-8\Delta^2 \mathrm{arcsin}(\omega/2\Delta)}{\pi\omega\sqrt{4\Delta^2-\omega^2}}\theta(2\Delta-\omega) + \left[\frac{8\Delta^2 \mathrm{arcosh}(\omega/2\Delta)}{\pi\omega\sqrt{\omega^2-4\Delta^2}} + i\frac{-4\Delta^2}{\omega\sqrt{\omega^2-4\Delta^2}}\right]\theta(\omega-2\Delta) \tag{44}$$

($\omega > 0$) and there are following relations between χ_i ($i = 0, 1, 2, 3$) and χ' at $\boldsymbol{q} = \boldsymbol{0}$: $1/p + \chi_2 = (\omega/2\Delta)^2\chi_3$, $\chi' = (-\omega/4\Delta)\chi_3$, $1/p + \chi_1 = [(\omega/2\Delta)^2 - 1]\chi_3$ and χ_0=0. Then, $-(1/p + \chi_2)\chi_3 + 4(\chi')^2 = 0$ at $\boldsymbol{q} = \boldsymbol{0}$.

With use of a relation $c_q = \pi\omega_p^2\tau/2D_\alpha q^2 \gg p$ (ω_p is the plasma frequency: $\omega_p^2 = 4\pi n_e e^2/m$ with $n_e = k_F^3/3\pi^2$ electron density and m the electron mass), Eqs. (23), (24), and (25) are approximately written as follows.

$$\Gamma_3(q) \simeq \frac{(1/p+\chi_2)/2}{-(1/p+\chi_2)\chi_3+4(\chi')^2}, \tag{45}$$

$$\Gamma_2(q) \simeq \frac{\chi_3/2}{-(1/p+\chi_2)\chi_3+4(\chi')^2}, \tag{46}$$

and

$$\Gamma'(q) \simeq \frac{\chi'}{-(1/p+\chi_2)\chi_3+4(\chi')^2}. \tag{47}$$

This expressions show that Γ_3,Γ_2, and Γ' are proportional to $1/x = 1/(D_\alpha q^2)$ because the denominator of these quantities vanishes at $\boldsymbol{q} = \mathbf{0}$. The above relations between χ_i ($i = 3, 2$) and χ' indicate that $\Gamma_3/\Gamma' = -\omega/2\Delta$ and $\Gamma_2/\Gamma' = -2\Delta/\omega$ at $\boldsymbol{q} = \mathbf{0}$. Then, in Eq. (43) the term containing $1/(x+\zeta^+_\epsilon+\zeta^+_{\epsilon-\omega})^2$ is proportional to the following equation:

$$\int d\omega \int dx\sqrt{x}\frac{\sum_{i=3,2}\Gamma_i(q)(h_i+g^+_\epsilon g^+_{\epsilon-\omega}+h'_i f^+_\epsilon f^+_{\epsilon-\omega})-2\Gamma'(q)(g^+_\epsilon f^+_{\epsilon-\omega}-f^+_\epsilon g^+_{\epsilon-\omega})}{(x+\zeta^+_\epsilon+\zeta^+_{\epsilon-\omega})^2}. \tag{48}$$

After the integration over x with use of $\Gamma \propto 1/x$, Eq. (48) is proportional to

$$\int d\omega\frac{\omega^2-4\Delta^2+(\omega^2+4\Delta^2)(g^+_\epsilon g^+_{\epsilon-\omega}-f^+_\epsilon f^+_{\epsilon-\omega})+4\omega\Delta(g^+_\epsilon f^+_{\epsilon-\omega}-f^+_\epsilon g^+_{\epsilon-\omega})}{(\zeta^+_\epsilon+\zeta^+_{\epsilon-\omega})^{3/2}}. \tag{49}$$

Both the numerator and the denominator of this expression vanish at $\omega = 2\epsilon$, and then the integration over ω results in a finite correction to the DOS. Therefore, by preserving the particle-number conservation, we obtain a finite result even when an additional singularity exists in the diffuson propagator in the superconducting state. (As for the case of the pole at $\omega = 0$ in Eq. (43), we obtain a finite result simply because $\Gamma_3 \propto \omega^2/x$, $\Gamma' \propto \omega/x$, and $h_2 + g^+_\epsilon g^-_{\epsilon-\omega} + h'_2 f^+_\epsilon f^-_{\epsilon-\omega} = 0$ at $\omega = 0$. The relation between Γ_3, Γ_2 and Γ' is irrelevant in this case.)

In the case of $\Gamma_{0,1}$, the long-range part $1/x$ is absent. As for the terms containing $\Gamma_{0,1}$, the integration over x is proportional to $1/(\zeta^+_\epsilon + \zeta^\pm_{\epsilon-\omega})^{1/2}$, which results in a finite value after the integration over ω is performed.

3.3 Numerical calculation

As discussed above, the dependence of $\rho'(\epsilon)$ on temperature is weak for $T \ll T_c$, and so we perform a numerical calculation at $T = 0$. We consider the superconducting gap at $T = 0$ as the unit of energy ($\Delta = 1$). p is determined by the gap equation.

The dependences of $\Gamma_i(q)$ and $\Gamma'(q)$ [Eqs. (45)$-$(47), (26) and (27)] on ω are shown in Fig. 2. (The value of α is implicitly included in $D_\alpha q^2$ and the result does not depend on α when the value of $D_\alpha q^2/\Delta$ is fixed.) $\mathrm{Im}\Gamma_i$ and $\mathrm{Im}\Gamma'$ take finite values above $\omega > 2\Delta$

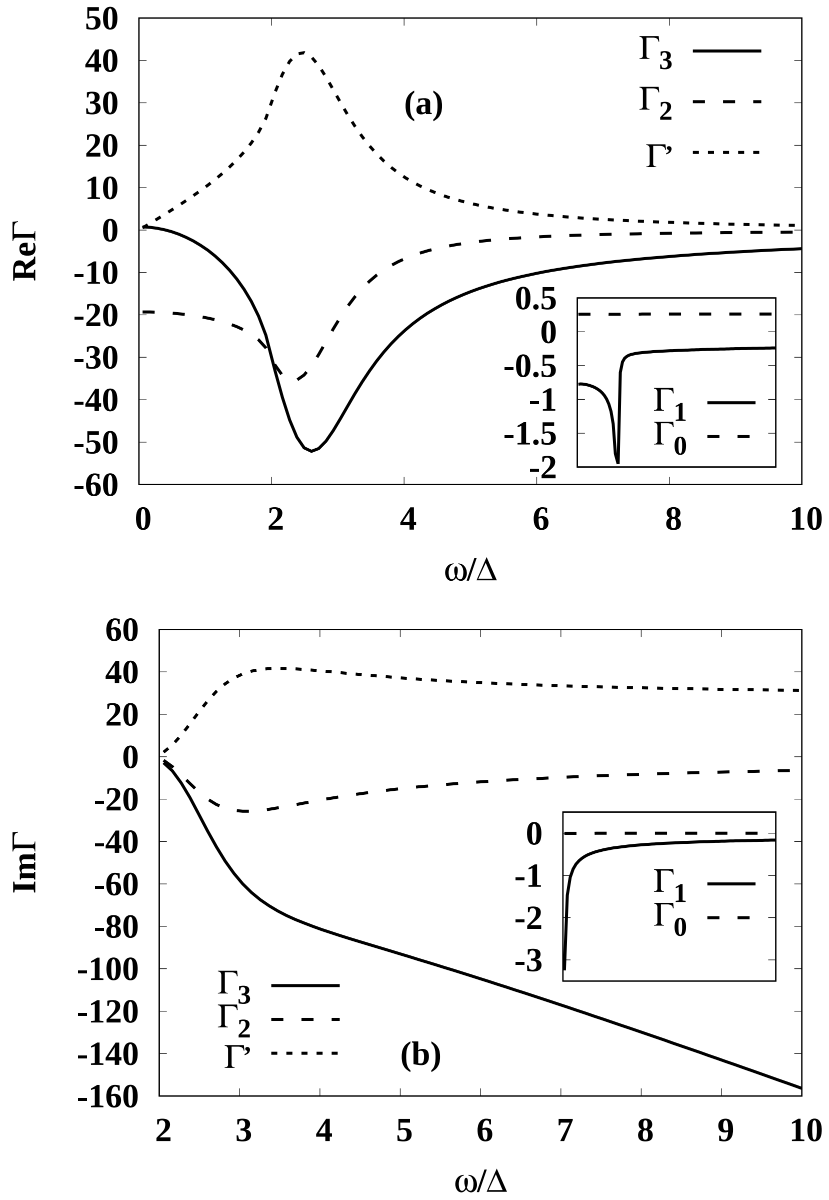

Fig. 2. The dependences of Γ_i and Γ' on ω at $T = 0$ and $D_\alpha q^2/\Delta = 0.055$. (a) The real part of Γ_i and Γ'. (b) The imaginary part of Γ_i and Γ'. The ranges of ω/Δ of the insets are the same as those of the main graphs.

owing to the finite excitation of quasiparticles across the superconducting gap. This leads to a peak in ReΓ around $\omega \simeq 2\Delta$. For $\omega \gg \Delta$, the dependence of Γ on ω should become close to that of the normal state. The large value of ImΓ_3 for $\omega \gg \Delta$ is related to $\Gamma_3(q) \simeq (\pi/4)(1 - i\omega/D_\alpha q^2)$ in the normal state obtained from Eq. (31). The sharp peak in Γ_1 around $\omega = 2\Delta$ indicates the existence of the amplitude mode. The density and phase fluctuations (Γ_3, Γ_2, and Γ'), however, are quantitatively predominant over

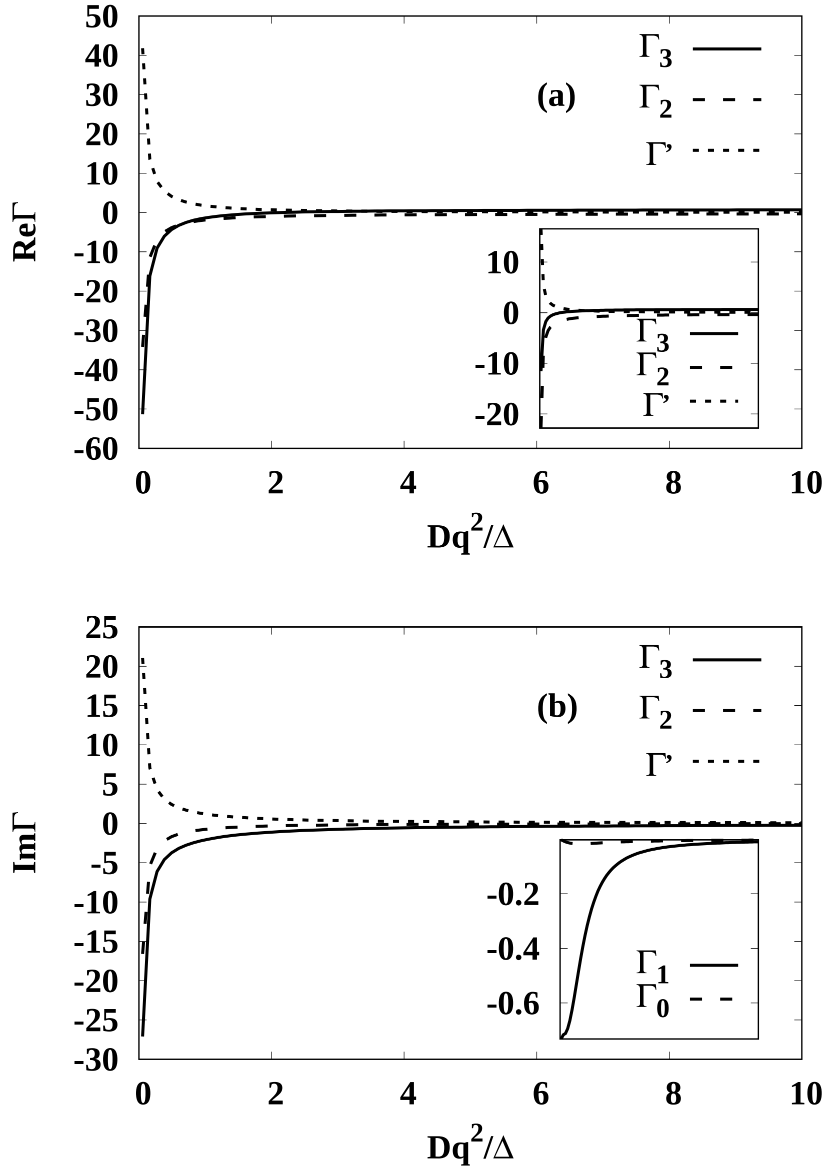

Fig. 3. The dependences of Γ_i and Γ' on $D_\alpha q^2$ at $T = 0$. (a) The real part of Γ_i and Γ' at $\omega/\Delta = 2.5$. The inset shows the results at $\omega/\Delta = 1.48$. (b) The imaginary part of Γ_i and Γ' at $\omega/\Delta = 2.5$. The ranges of $D_\alpha q^2/\Delta$ of the insets are the same as those of the main graphs.

$\Gamma_{1,0}$. These large values come from the long-range part ($\propto 1/q^2$).

The dependences of $\Gamma_i(q)$ and $\Gamma'(q)$ [Eqs. (45)−(47), (26) and (27)] on $D_\alpha q^2$ are shown in Fig. 3. Γ_3, Γ_2 and Γ' are proportional to $1/D_\alpha q^2$. The results show that these three terms (the density and phase fluctuations) are quantitatively comparable to each other. This validates the argument about diffuson in the previous subsection.

Next, we calculate the correction to the DOS numerically. From Eq. (43), we write the correction to DOS as follows:

$$\rho'(\epsilon) = \frac{\rho_0|\epsilon|}{\sqrt{\epsilon^2 - \Delta^2}}\delta\rho_\epsilon. \tag{50}$$

In the case of the normal state, $\delta\rho_\epsilon = \rho'(\epsilon)/\rho_0$ from Eqs. (34) -(36). The calculation in the superconducting state is performed at $T = 0$ as noted above. In the case of the normal state, the superconducting fluctuation depends on the temperature. We fix $T = 1.1T_C$ in Eq. (30) and assume $T = 0$ in other terms. We take $|\omega| < 1/\tau$ and $x = D_\alpha q^2 < 4/\tau$ as the range of integrations in Eqs. (35), (36) and (43). The energy dependence of $\delta\rho_\epsilon$ is mainly determined by the low-energy part $|\omega|, x \ll 1/\tau$. When we change the upper limits of $|\omega|$ and x, only the magnitude of $|\delta\rho_\epsilon|$ is shifted. We consider the weak-coupling case for the interaction between electrons and phonons. This interaction is taken to vanish outside the cutoff frequency (ω_c), and then $\Gamma_i(q), \Gamma'(q) \neq 0$ ($i = 0, 1, 2$) only for $|\epsilon|, |\epsilon-\omega| < \omega_c$ [$\Gamma_3(q)$ is finite outside this region.] We take $\omega_c = 10\Delta$ in the numerical calculation. We specify the relation between $\alpha = 1/2\tau$ and $k_F l$ in Eqs. (35), (36) and (43) by putting $k_F l/2\tau = E_F = 300\Delta$ (E_F is the Fermi energy).

The calculated results of the correction to the DOS are shown in Fig. 4. "SC" and "N" are the results calculated in the superconducting state and the normal state, respectively. "N_0" is the calculated result with only the term $\Gamma_3(q)$ included in Eq. (36). The dependence of $\delta\rho_\epsilon$ on ϵ changes slightly with increasing α, and it is written as $\delta\rho_\epsilon \propto \sqrt{\epsilon}$ for "N_0". As for the dependence of the magnitude of $\delta\rho_\epsilon$ on α, $\delta\rho_\epsilon \propto 1/(k_F l)^2$ holds in both the superconducting and the normal states. This is related to the ϵ-dependence of $\delta\rho_\epsilon$ because the equation

$$\rho'_{cl}(\epsilon) \simeq \rho_0 \frac{-3\sqrt{3\tau/2}}{(2k_F l)^2} \int_\epsilon^{1/\tau} d\omega \frac{1}{\sqrt{\omega}} \propto \frac{\sqrt{\tau}}{(k_F l)^2}\left(-\frac{1}{\sqrt{\tau}} + \sqrt{\epsilon}\right) \tag{51}$$

is derived from Eq. (36).

The ϵ-dependences of $\delta\rho_\epsilon$ are not exactly written as $\delta\rho_\epsilon \propto \sqrt{\epsilon}$ for "SC" and "N". The result for "SC" shows that a dip structure appears around $\epsilon = 3\Delta$. This structure is resulted from the peak in $\Gamma(q)$ around $\omega \simeq 2\Delta$. The reason for the overall suppression in "SC" as compared to "N" is the enhancement of Γ_2 and Γ' owing to the coupling of the phase fluctuation to the density fluctuation. The result for "N" shows that the superconducting fluctuation suppresses the DOS at low energy. The $\delta\rho_\epsilon$ values of "SC" and "N" approach that of "N_0" at high energy owing to the weakening of the superconducting correlation for $\epsilon \gg \Delta$.

The difference in magnitude between Γ_i and Γ' shown in Fig. 2 is directly reflected in

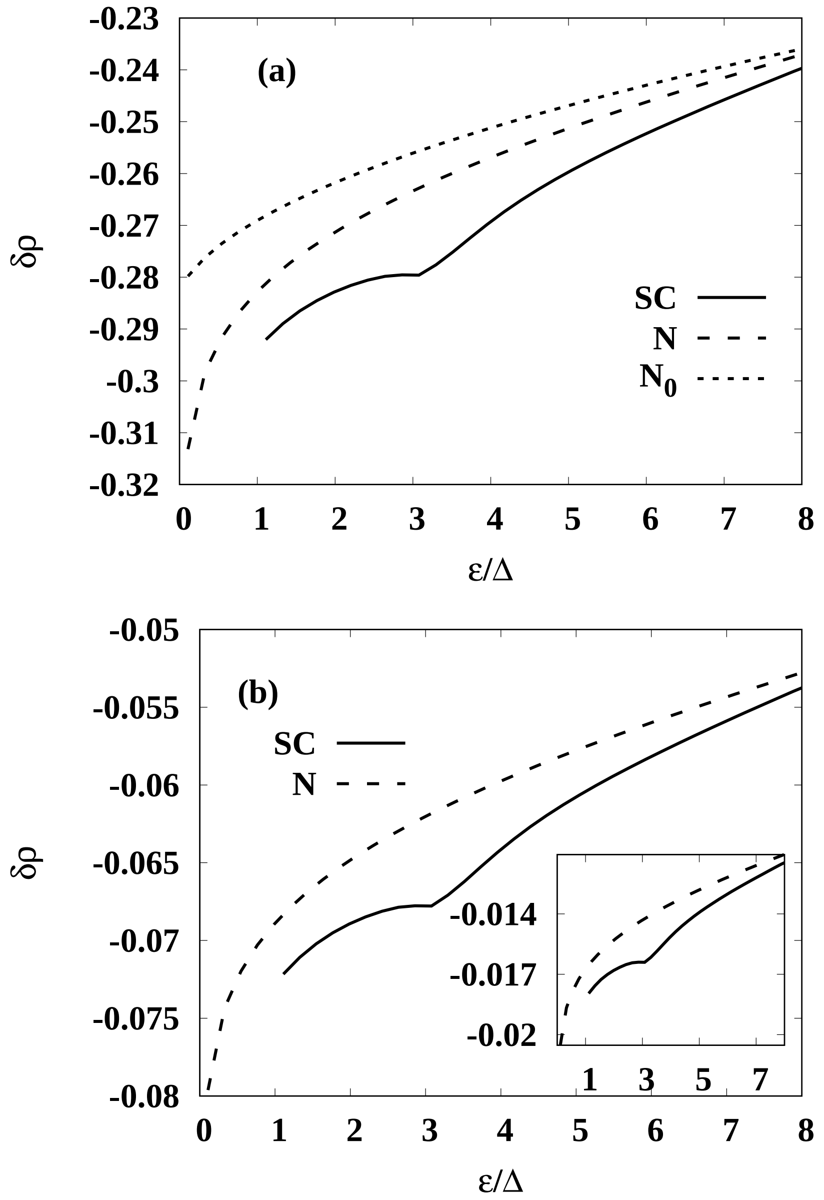

Fig. 4. The dependences of the correction to the DOS on ϵ at $T = 0$. (a) $\alpha/\Delta = 120$ ($k_F l = 2.5$). (b) $\alpha/\Delta = 60$ ($k_F l = 5.0$). The inset shows the result for $\alpha/\Delta = 30$ ($k_F l = 10.0$). The meanings of "SC", "N" and "N_0" are given in the text.

$\delta\rho_\epsilon$. $\delta\rho_\epsilon$ in the superconducting state is decomposed into several terms, and the results are shown in Fig. 5. The decomposition is done according to Γ_i and Γ' contained in Eq. (43). For example, "$3, 2, '$" in Fig. 5 represents the contribution from Γ_3, Γ_2 and Γ' to $\delta\rho_\epsilon$. The calculated results show that the phase and density fluctuations majorly contribute to $\delta\rho_\epsilon$ because they contain the long-range part ($\propto 1/q^2$). The contribution from the amplitude fluctuation is small, as illustrated in Fig. 2.

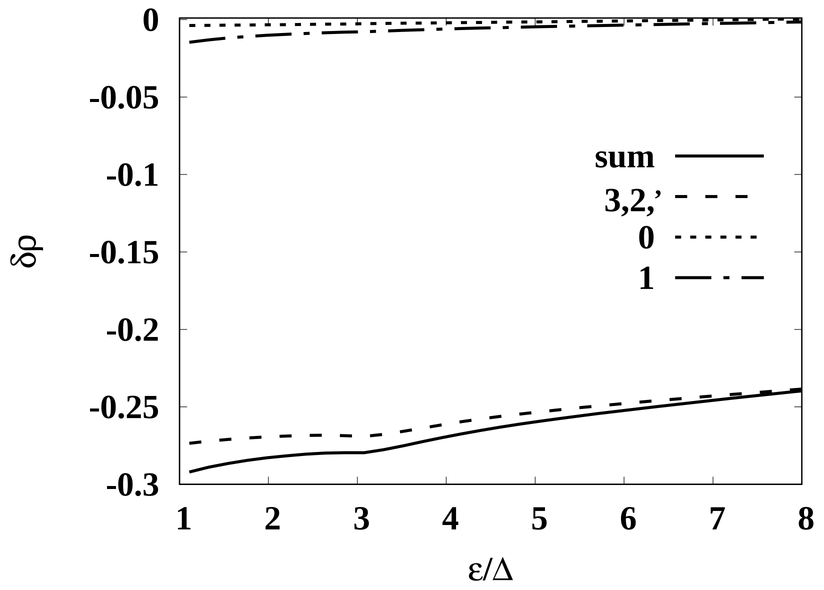

Fig. 5. The decomposition of $\delta\rho_\epsilon$ into several terms according to Γ_i and Γ' contained in $\delta\rho_\epsilon$. "$3, 2, '$", "0" and "1" correspond to the suffixes of Γ_i and Γ'. "sum" indicates the summation of these three quantities. $\alpha = 120\Delta$ ($k_F l = 2.5$) and $T = 0$.

Equation (43) seemingly includes a divergence proportional to $1/\sqrt{\epsilon^2 - \Delta^2}$ in $\delta\rho_\epsilon$. To clarify the reason for the absence of this divergence in Fig. 3, we decompose Eq. (43) as follows:

$$\rho'(\epsilon) = \frac{\rho_0|\epsilon|}{\sqrt{\epsilon^2 - \Delta^2}} \left(\delta\rho_\epsilon^{sf} + \delta\rho_\epsilon^{cl}\right) \tag{52}$$

with

$$\delta\rho_\epsilon^{sf} = \frac{-3\sqrt{3\tau}}{(2\pi k_F l)^2} \int d\omega \int dx \sqrt{x}$$
$$\times \operatorname{Im}\Big\{ 2\coth\left(\frac{\omega}{2T}\right) \frac{\operatorname{Im}[\Gamma_i(q)](h_i + g_\epsilon^+ g_{\epsilon-\omega}^+ + h_i' f_\epsilon^+ f_{\epsilon-\omega}^+) + 2\operatorname{Im}[\Gamma'(q)](f_\epsilon^+ g_{\epsilon-\omega}^+ - g_\epsilon^+ f_{\epsilon-\omega}^+)}{(x + \zeta_\epsilon^+ + \zeta_{\epsilon-\omega}^+)^2}$$
$$+ \tanh\left(\frac{\epsilon - \omega}{2T}\right) \frac{\Gamma_i(q)(h_i + g_\epsilon^+ g_{\epsilon-\omega}^+ + h_i' f_\epsilon^+ f_{\epsilon-\omega}^+) + 2\Gamma'(q)(f_\epsilon^+ g_{\epsilon-\omega}^+ - g_\epsilon^+ f_{\epsilon-\omega}^+)}{(x + \zeta_\epsilon^+ + \zeta_{\epsilon-\omega}^+)^2} \Big\} \tag{53}$$

and

$$\delta\rho_\epsilon^{cl} = \frac{-3\sqrt{3\tau}}{(2\pi k_F l)^2} \int d\omega \int dx \sqrt{x}$$
$$\times \operatorname{Im}\Big\{ \tanh\left(\frac{\epsilon - \omega}{2T}\right) \frac{\Gamma_i(q)(h_i + g_\epsilon^+ g_{\epsilon-\omega}^- + h_i' f_\epsilon^+ f_{\epsilon-\omega}^-) + 2\Gamma'(q)(f_\epsilon^+ g_{\epsilon-\omega}^- - g_\epsilon^+ f_{\epsilon-\omega}^-)}{(x + \zeta_\epsilon^+ + \zeta_{\epsilon-\omega}^-)^2} \Big\}. \tag{54}$$

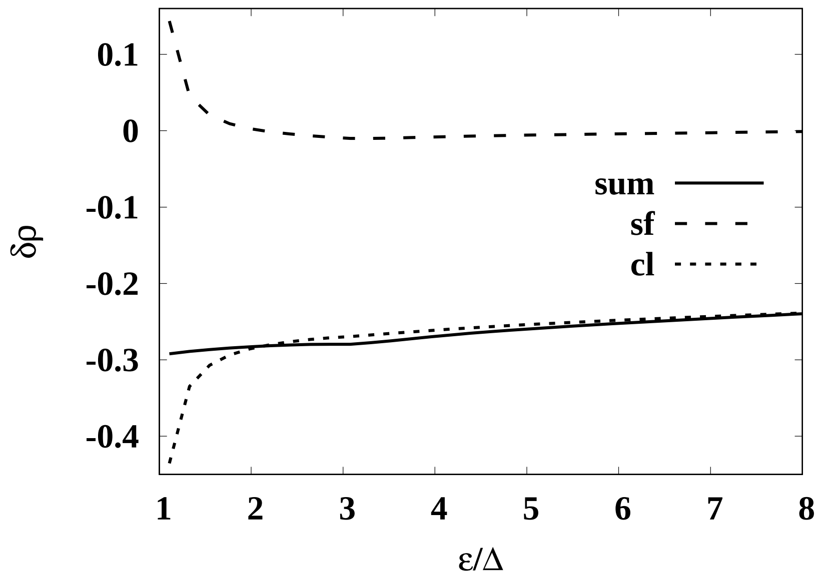

Fig. 6. The decomposition of $\delta\rho_\epsilon$. "sf" and "cl" indicate $\delta\rho_\epsilon^{sf}$ and $\delta\rho_\epsilon^{cl}$, respectively. "sum" indicates the summation of these two quantities. $\alpha = 120\Delta$ ($k_F l = 2.5$) and $T = 0$.

The calculated results for these quantities are shown in Fig. 6. When $\Delta = 0$, Eqs. (53) and (54) reduce to Eqs. (35) and (36) (except for the factor ρ_0), respectively. Both $\delta\rho_\epsilon^{sf}$ and $\delta\rho_\epsilon^{cl}$ include the effects of the superconducting fluctuation and the Coulomb interaction in the case of $\Delta \neq 0$. $\delta\rho_\epsilon^{sf(cl)}$ includes only the "retarded (advanced)" quantities ($\zeta_{\epsilon-\omega}^{+(-)}$, $g_{\epsilon-\omega}^{+(-)}$ and $f_{\epsilon-\omega}^{+(-)}$). The calculated results show that the absence of the divergence proportional to $1/\zeta_\epsilon^+ = i/\sqrt{\epsilon^2 - \Delta^2}$ in $\delta\rho_\epsilon$ is caused by the cancellation between the retarded and the advanced parts [terms proportional to $1/(x + \zeta_\epsilon^+ + \zeta_{\epsilon-\omega}^+)^2$ and $1/(x + \zeta_\epsilon^+ + \zeta_{\epsilon-\omega}^-)^2$].

The DOS with the correction included is written as follows:

$$\rho(\epsilon) = \frac{\rho_0|\epsilon|}{\sqrt{\epsilon^2 - \Delta^2}} + \rho'(\epsilon) = \frac{\rho_0|\epsilon|(1 + \delta\rho_\epsilon)}{\sqrt{\epsilon^2 - \Delta^2}}. \tag{55}$$

The calculated result of this expression is shown in Fig. 7. In the normal state, $\rho(\epsilon) = \rho_0(1 + \delta\rho_\epsilon)$. The result shows that $\rho(\epsilon)$ increases with increasing ϵ for large α. For small α, $\rho(\epsilon)$ decreases as $|\epsilon|/\sqrt{\epsilon^2 - \Delta^2}$ because of the small values of $\delta\rho_\epsilon$. This indicates that, although the ϵ-dependence of $\delta\rho_\epsilon$ is almost independent of α, as shown in Fig. 4, the increasing DOS with $|\epsilon|$ is observable only for large α.

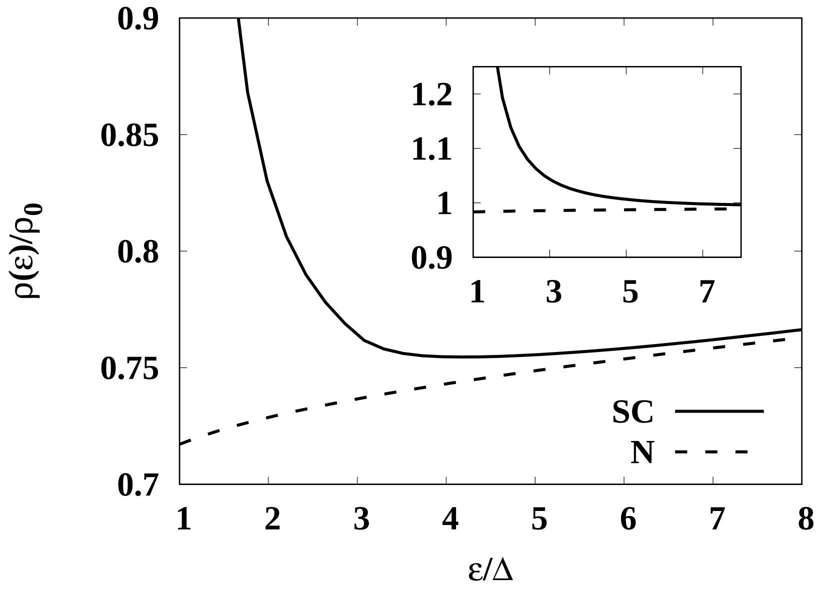

Fig. 7. The DOS with the correction included. $\alpha = 120\Delta$ ($k_F l = 2.5$). "SC" and "N" indicate the result for the superconducting and the normal state, respectively. The inset shows the result for $\alpha = 30\Delta$ ($k_F l = 10.0$).

4. Summary and Discussion

In this study, we calculated the correction to the DOS perturbatively. The correction term is given by the Coulomb interaction and the electron-phonon interaction, with vertices of these interactions modified by the impurity scattering. The modification enhances these interactions at low energy. The energy dependence of the correction to DOS in the superconducting state is different from that in the normal state, and a dip structure appears at low energy. This structure is caused by the interaction which has a peak at about twice the energy of the superconducting gap. (The dip structure in the one-particle spectrum is also observed in cuprates, but its origin is different.[23–25])

There are two differences between the superconducting state and the normal state. First, the diffuson is modified because the opening of the superconducting gap changes the dispersion of quasiparticles. This gives rise to another pole in the diffuson propagator, and this pole is treated correctly by including the coupling of the density and phase fluctuations. Second, the correction to DOS does not affect the gap-edge singularity in the superconducting state. This is because the cancellation between the retarded and advanced parts occurs around the gap edge. In the normal state, the superconducting fluctuation and the Coulomb interaction separately contribute to the retarded and advanced parts, respectively. In the superconducting state we cannot treat them separately and need to include both parts simultaneously in the correction to DOS.

Regarding the validity of perturbation expansion, if we consider the perturbation expansion in the case of the scattering by nonmagnetic impurities, the correction to DOS is proportional to $\mathrm{Im}\sum_{\boldsymbol{k},k'}\mathrm{Tr}[\hat{G}_k\hat{\tau}_3\hat{G}_{k'}\hat{\tau}_3\hat{G}_k]=0$. The nonmagnetic impurities do not affect the DOS in the Born approximation. In contrast, for paramagnetic impurities, the correction to DOS is proportional to $\mathrm{Im}\sum_{\boldsymbol{k},k'}\mathrm{Tr}[\hat{G}_k\hat{\tau}_0\hat{G}_{k'}\hat{\tau}_0\hat{G}_k]\propto\Delta^2|\epsilon|/(\epsilon^2-\Delta^2)^{3/2}$. This means that the perturbation expansion is invalid around $|\epsilon|\simeq\Delta$, and the gap edge in the DOS changes qualitatively.[26,27] The calculation in this paper shows that the correction to DOS does not diverge around the gap edge. This indicates that the perturbation expansion is valid within our approximations.

We calculated the Fock term with its vertices modified by diffuson (for example, Fig. 3 (a) in Ref. 6, with the wavy line in this figure replaced by the Coulomb interaction and the superconducting fluctuation in our calculation). It is possible to consider other types of diagrams. For example, these are the Fock terms with its vertices modified by Cooperon and the Hartree term (Figs. 3 (b)−(d) in Ref. 6). The correction to DOS by the Fock term with Cooperon is proportional to $-\mathrm{Im}\sum_{\boldsymbol{k},q}\mathrm{Tr}[\hat{G}_k\cdots\sum_{k_1,k_2}\Gamma_{k_1-k_2}\hat{G}_{k_1}\hat{\tau}_3\hat{G}_{k_2}\cdots\hat{G}_{q-k}\cdots\hat{G}_{q-k_1}\hat{\tau}_3\hat{G}_{q-k_2}\cdots\hat{G}_k]$. The singular part $\Gamma_q\propto 1/\boldsymbol{q}^2$ (which majorly contributes to the correction to DOS in our calculation) is weakened when the summations are performed. Thus, we can omit this type of diagram. There is a similar term in the case of the Hartree diagram modified by diffuson or Cooperon. (In the case of the Fock term modified by diffuson, $-\mathrm{Im}\sum_{\boldsymbol{k},q}\mathrm{Tr}[\hat{G}_k\cdots\sum_{k_1,k_2}\Gamma_q\hat{G}_{k_1}\hat{\tau}_3\hat{G}_{k_1-q}\cdots\hat{G}_{k-q}\cdots\hat{G}_{k_2-q}\hat{\tau}_3\hat{G}_{k_2}\cdots\hat{G}_k]$.)

This study considers the case of low temperatures ($T\ll\Delta$). The superconducting gap Δ was taken as the unit of energy, and we did not consider the interaction effect on the superconducting gap. When the temperature is comparable to the superconducting gap, the self-consistency through the gap equation becomes important.

Finally, we comment on the possibility of observing a dip structure in experiments. Experimentally, it is known that the superconducting state becomes inhomogeneous with decreasing k_Fl,[17] and the one-particle spectrum is averaged over these inhomogeneous states. (There are also theoretical studies on inhomegeneities in superconductors without Coulomb interaction.[28,29] In addition, the perturbative calculation should be modified for small values of k_Fl near the insulating state, and the renormalization-group method[30] will be required.) Thus, it is difficult to observe the dip structure in the case of large values of α. Figures 4 and 7 show, however, that the dip structure is possibly observed even for small values of α ($k_F l \gg 1$, but in the dirty limit $\Delta\tau \ll 1$) when the overall factor $|\epsilon|/\sqrt{\epsilon^2 - \Delta^2}$ is removed. The dip structure originates from the interactions in the superconducting state, and therefore the difference between our calculation and the calculations using the Coulomb interaction and diffuson of the normal state[18,19] appears in this quantity.

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