

# Damping Rate of Josephson Plasmons by Photoexcited Quasiparticles in Small Tunnel Junctions

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We investigate the quasiparticle excitation by an electromagnetic field in a small tunnel junction and its effect on the damping rate of Josephson plasmons. By the time-dependent path integral method, we formulate the interaction between Josephson plasmons and external fields. Because both quantities are related to each other by the tunneling term, a microscopic derivation of the interaction is important in performing quantitative calculations consistently. We derive an analytical expression for the damping rate in the dispersive region where the oscillating electric field is detuned from Josephson plasmons, and then perform numerical calculations to obtain the relaxation time quantitatively. The result shows that by exciting quasiparticles, the electromagnetic field brings about the decay of Josephson plasmons even in the case that its frequency is lower than twice the superconducting gap. This indicates that there is another source of relaxation besides the spontaneous emission and Purcell effect.

## 1. Introduction

The Josephson plasmon is an excitation that exists in the junction between two superconductors. Its excitation energy is lower than the superconducting gap, and by utilizing its long coherence time, this excitation can be used as a superconducting quantum bit (a quantum two-level system).<sup>1)</sup>

A long relaxation time is crucial in manipulating the superconducting qubit, but the excited state decays by various sources<sup>2-5)</sup> because of its macroscopic nature. The major sources for its decay are considered to be extrinsic, and this speculation comes from the fact that the lifetime is still increasing,<sup>6)</sup> and the intrinsic decay owing to the thermally excited quasiparticles is exponentially small at low temperatures.<sup>7)</sup>

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Although there are several extrinsic effects, the inevitable effect is caused by the interaction with the electromagnetic field. This is because the manipulation of the superconducting qubit is often carried out by making use of microwave photons.<sup>8)</sup> This especially applies to the case of the transmon qubit,<sup>9)</sup> which has a long relaxation time<sup>6)</sup> by setting the qubit in the three-dimensional architecture.<sup>10)</sup>

There are two effects of the electromagnetic field on the damping rate: the spontaneous emission and the Purcell effect.<sup>11)</sup> The former is the effect of the vacuum fluctuation and is inevitable. The latter originates from the existence of the cavity, and this effect is dependent on the shape and quality of materials. Theoretically, both effects are treated by introducing the bilinear coupling term, which is often used to describe the interaction with environments.

In this study, we investigate another effect of the electromagnetic field, and find that quasiparticles excited by external fields can be a source of the decay of Josephson plasmons by making a quantitative estimation. In the three-dimensional structure, there are several modes in the cavity,<sup>12,13)</sup> and it is possible that quasiparticles also are excited by cavity modes. This is different from the Purcell effect, in which the damping rate of external fields is phenomenologically taken into account, and this leads to the decay of excited states.

In Sect. 2, the action for Josephson plasmons with external fields included is given, and after specifying the electric field, the Dyson equation is derived in the dispersive region where the electromagnetic modes are detuned from Josephson plasmons. Several approximations hold in this region. By using this formulation, the damping rate and the energy shift by external fields are derived and numerically computed in Sect. 3. The quantitative calculation of the damping rate shows that considerable decay occurs even at frequencies lower than twice the superconducting gap. In Sect. 4, other effects of the electromagnetic field are discussed.

## 2. Phase Fluctuation under External Fields

### 2.1 *Effective action*

The effective action for the out-of-phase mode of the phase fluctuation at the interface between the superconductor and the insulating barrier (Josephson plasmons) is derived in Ref. 14, in which the Matsubara frequency is used. The extension to describe the time-dependent phenomena is carried out by exploiting the method described in

Ref. 15. The result is  $iS_{\text{eff}} = iS_0 + iS_1 + \sum_{l=2}^{\infty} iS_{2l} + \sum_{l=1}^{\infty} iS_{2l+1}$ . Here,

$$iS_0 = \frac{1}{2} \int d\omega \begin{pmatrix} \phi_{-\omega}^c & \phi_{-\omega}^q \end{pmatrix} \hat{m} \begin{pmatrix} \phi_{\omega}^c \\ \phi_{\omega}^q \end{pmatrix},$$

$$iS_1 = \int d\omega \frac{-E_J(\omega)}{2i} 2edA_{\omega}^c \phi_{-\omega}^q,$$

$$iS_{2l} = \frac{1}{2E_C i} \left( \frac{-1}{4} \right)^{l-1} \int \frac{d\omega_1 \cdots d\omega_{2l}}{(2\pi)^{l-1}} \left( \prod_{i=1}^{2l-1} \phi_{\omega_i}^c \right) \phi_{\omega_{2l}}^q \delta(\bar{\omega}_{1,2l}) \\ \times \left[ \sum_{n=1}^l \frac{g_{00}(\bar{\omega}_{1,2n-1})}{(2n-1)!(2l-2n)!} + \sum_{n=1}^{l-1} \frac{-g_{33}(\bar{\omega}_{1,2n})}{(2n)!(2l-2n-1)!} + \frac{-g_{33}(0)}{(2l-1)!} \right],$$

and

$$iS_{2l+1} = \frac{1}{2E_C i} \left( \frac{-i}{2} \right)^{2l-1} \int \frac{d\omega_1 \cdots d\omega_{2l+1}}{(2\pi)^{l-1/2}} \left( \prod_{i=1}^{2l} \phi_{\omega_i}^c \right) \phi_{\omega_{2l+1}}^q \delta(\bar{\omega}_{1,2l+1}) \left[ \sum_{n=1}^{2l} \frac{g_{30}(\bar{\omega}_{1,n})}{(n)!(2l-n)!} + \frac{g_{30}(0)}{(2l)!} \right]$$

( $\bar{\omega}_{1,2l} := \sum_{i=1}^{2l} \omega_i$ ). The definitions of quantities in the above action are as follows.

$\phi_{\omega}^{c,q} = \int \frac{dt}{\sqrt{2\pi}} \phi_t^{c,q} e^{i\omega t}$  indicates the phase fluctuation at the interface with the use of the

transformation  $\begin{pmatrix} \phi_t^c \\ \phi_t^q \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \phi_t^+ \\ \phi_t^- \end{pmatrix}$ , in which the superscript  $+(-)$  indicates

the forward (backward) direction in time.<sup>15)</sup> We omit the components of finite wave

numbers because these excitation energies are high in a small tunnel junction as dis-

cussed in Sect. 5 of Ref. 14  $\phi_{\omega}^c = \phi_{\omega}^c - 2edA_{\omega}^c$  includes the classical external field  $A_{\omega}^c$  (we

do not consider the quantum fluctuation  $A_{\omega}^q$  in this paper), and  $d$  is the thickness of the

insulating barrier of the Josephson junction. The propagator of the phase fluctuation is

$D_{\omega}^{cq0} = \frac{2E_C i}{(\omega+i0)^2 - \omega_0^2}$  (this is a retarded Green's function), and by using this quantity, the

matrix in  $iS_0$  is written as  $\hat{m} = \begin{pmatrix} 0 & -(D_{\omega}^{qc0})^{-1} \\ -(D_{\omega}^{cq0})^{-1} & -\coth \frac{\omega}{2T} [(D_{\omega}^{cq0})^{-1} - (D_{\omega}^{qc0})^{-1}] \end{pmatrix}$  (here,

$D_{\omega}^{qc0} = D_{-\omega}^{cq0}$ ).  $\omega_0$  is the Josephson plasma frequency and  $\omega_0^2 = E_C E_J(\omega)$  ( $E_C$  and  $E_J$

are the charging and Josephson coupling energy, respectively, and the definition of  $E_C$

is given in Ref. 16).  $E_J$  also includes the damping rate of the Josephson plasmons by

quasiparticle excitations, and its expression is written as  $E_J(\omega) = [g_{00}(\omega) - g_{33}(0)]/E_C$

with

$$\frac{g_{ij}(\omega)}{N^2} = E_C t'^2 \frac{1}{N^2} \left( \frac{2}{N_z + 1} \right)^2 \sum_{k,\zeta,\zeta'} \int \frac{d\omega}{2\pi i} \text{Tr}[\hat{G}_{k\zeta}^{L+}(\epsilon+\omega) \hat{\tau}_i \hat{G}_{k\zeta'}^{RK}(\epsilon) \hat{\tau}_j + \hat{G}_{k\zeta}^{LK}(\epsilon+\omega) \hat{\tau}_i \hat{G}_{k\zeta'}^{R-}(\epsilon) \hat{\tau}_j].$$

Here,  $\hat{G}_{k\zeta}^{L\pm}(\epsilon) = \frac{1}{(\epsilon \pm i0)^2 - \xi_{k\zeta}^2 - |\Delta_L|^2} \begin{pmatrix} \epsilon \pm i0 + \xi_{k\zeta} & -\Delta_L \\ -\Delta_L^* & \epsilon \pm i0 - \xi_{k\zeta} \end{pmatrix}$  and  $\hat{G}_{k\zeta}^{LK}(\epsilon) = \tanh \frac{\epsilon}{2T} [\hat{G}_{k\zeta}^{L+}(\epsilon) - \hat{G}_{k\zeta}^{L-}(\epsilon)]$  are the Green's function of electrons [the superscript  $+(-)$  indicates the retarded (advanced) Green's function and  $L(R)$  the superconductor at the left (right) of the junction].  $\Delta_L = \Delta e^{i\phi_L}$  is the superconducting gap of the left superconductor with amplitude  $\Delta$  and static phase  $\phi_L$  (we assume that the amplitudes are the same in the left and right superconductors).  $t'$  is the transfer integral across the junction and  $N^2$  is the number of sites at the interface (we set the lattice constant  $a = 1$ ).  $\zeta$  indicates the degree of freedom perpendicular to the interface and  $N_z$  the number of sites in this direction.<sup>14)</sup>  $\hat{\tau}_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$  and  $\hat{\tau}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ . Finally, by transforming the variables  $\begin{pmatrix} \phi_\omega^c & \phi_\omega^q \end{pmatrix}$  into  $\begin{pmatrix} \phi_\omega'^c & \phi_\omega^q \end{pmatrix}$ , the effective action is rewritten as  $iS'_{\text{eff}} = iS'_0 + iS'_1 + \sum_{l=2}^{\infty} iS_{2l} + \sum_{l=2}^{\infty} iS_{2l-1}$ . Here,  $iS'_0 = \frac{1}{2} \int d\omega \begin{pmatrix} \phi_{-\omega}'^c & \phi_{-\omega}^q \end{pmatrix} \hat{m} \begin{pmatrix} \phi_\omega'^c \\ \phi_\omega^q \end{pmatrix}$  and  $iS'_1 = \int d\omega A'_\omega \phi_{-\omega}^q$  with  $A'_\omega := \frac{-(\omega+i0)^2}{2E_C i} 2edA_\omega^c$ .

## 2.2 Dyson equation under oscillating external fields

The propagator with the external field included is defined as

$$D_{\omega,\omega'}^{cq} = \langle \phi_\omega'^c \phi_{-\omega'}^q \rangle = \frac{\int \mathcal{D}[\phi'^c, \phi^q] \phi_\omega'^c \phi_{-\omega}^q \exp[iS'_{\text{eff}}]}{\int \mathcal{D}[\phi'^c, \phi^q] \exp[iS'_{\text{eff}}]},$$

which satisfies the equation

$$D_{\omega,\omega'}^{cq} = D_\omega^{cq0} \delta(\omega - \omega') + D_\omega^{cq0} \int d\omega_1 \Pi_{\omega,\omega_1} D_{\omega_1,\omega'}^{cq}.$$

$\Pi_{\omega,\omega'}$  is the self-energy that describes the effect of the external field:

$$\begin{aligned} \Pi_{\omega,\omega'} &= \frac{1}{2E_C i} \sum_{l=1}^{\infty} \left( \frac{-1}{4} \right)^l \int \frac{d\omega_1 \cdots d\omega_{2l}}{(2\pi)^l} \left( \prod_{i=1}^{2l} A'_{\omega_i} D_{\omega_i}^{cq0} \right) \delta(\bar{\omega}_{1,2l} + \omega' - \omega) \\ &\quad \times \left[ \frac{g_{00}(\omega') - g_{33}(0)}{(2l)!} + \sum_{n=1}^l \frac{g_{00}(\omega' + \bar{\omega}_{1,2n}) - g_{33}(\bar{\omega}_{1,2n})}{(2n)!(2l-2n)!} + \sum_{n=1}^l \frac{g_{00}(\bar{\omega}_{1,2n-1}) - g_{33}(\omega' + \bar{\omega}_{1,2n-1})}{(2n-1)!(2l-2n+1)!} \right] \\ &+ \frac{1}{2E_C i} \sum_{l=1}^{\infty} \left( \frac{-i}{2} \right)^{2l-1} \int \frac{d\omega_1 \cdots d\omega_{2l-1}}{(2\pi)^{l-1/2}} \left( \prod_{i=1}^{2l-1} A'_{\omega_i} D_{\omega_i}^{cq0} \right) \delta(\bar{\omega}_{1,2l-1} + \omega' - \omega) \\ &\quad \times \left[ \frac{g_{30}(0) + g_{30}(\omega')}{(2l-1)!} + \sum_{n=1}^{2l-1} \frac{g_{30}(\bar{\omega}_{1,n}) + g_{30}(\omega' + \bar{\omega}_{1,n})}{(n)!(2l-1-n)!} \right]. \end{aligned}$$

Here, we introduce an approximation and calculate only the tree-type graph (some part of the diagrams are shown below). This does not include the loop correction that becomes small with the expansion parameter  $\sqrt{E_C/E_J}$ .<sup>16)</sup> (We consider the case of

$E_C \ll E_J$  such as a transmon.<sup>9)</sup> Although the loop correction is important in the resonant region owing to the resulting energy shifts, the above approximation is justified in the dispersive region.

We consider an oscillating electric field across the junction as the external field:  $E_t = E \cos(\Omega t)$ , and then the vector potential is  $A_t^c = -\frac{E}{\Omega} \sin(\Omega t)$ . The dispersive region is written as  $|\Omega - \omega_0| \gg edE$  with this notation. By using this quantity and  $a_{\pm} := \frac{\pm(\Omega \pm i0)^2}{(\Omega \pm i0)^2 - \omega_0^2} \prod_{i=1}^{2l} A'_{\omega_i} D_{\omega_i}^{cq0} = (-2\pi)^l \left(\frac{edE}{\Omega}\right)^{2l} \prod_{i=1}^{2l} [a_+ \delta(\omega_i - \Omega) + a_- \delta(\omega_i + \Omega)]$ , we obtain

$$\begin{aligned} \Pi_{\omega, \omega'} = \frac{1}{2E_C i} \sum_{n, m=-\infty}^{\infty} \bigg\{ & [J_{2n}(\varepsilon) J_{2m}(\varepsilon) - \delta_{n,0} \delta_{m,0}] [g_{00}(\omega' + \Omega_{2n}) - g_{33}(\Omega_{2n})] \delta(\omega - \omega' - \Omega_{2n} - \Omega_{2m}) \\ & + J_{2n-1}(\varepsilon) J_{2m+1}(\varepsilon) [g_{00}(\Omega_{2n-1}) - g_{33}(\omega' + \Omega_{2n-1})] \delta(\omega - \omega' - \Omega_{2n-1} - \Omega_{2m+1}) \\ & + J_{2n}(\varepsilon) J_{2m-1}(\varepsilon) [g_{30}(\Omega_{2n}) + g_{30}(\omega' + \Omega_{2n})] \delta(\omega - \omega' - \Omega_{2n} - \Omega_{2m-1}) \\ & + J_{2n-1}(\varepsilon) J_{2m}(\varepsilon) [g_{30}(\Omega_{2n-1}) + g_{30}(\omega' + \Omega_{2n-1})] \delta(\omega - \omega' - \Omega_{2n-1} - \Omega_{2m}) \bigg\}. \end{aligned} \quad (1)$$

Here,  $\varepsilon := edE\alpha/\Omega$ ,  $\alpha := \Omega^2/(\Omega^2 - \omega_0^2)$ ,  $\Omega_n := n\Omega$ , and  $J_n(z)$  is the Bessel function of the first kind. We rewrite the above equation as  $\Pi_{\omega, \omega'} = \frac{1}{2E_C i} \sum_{k=-\infty}^{\infty} \Pi_{\omega'}^{(k)} \delta(\omega - \omega' - \Omega_k)$  with  $\Pi_{\omega'}^{(2k)} = \sum_{n=-\infty}^{\infty} \left\{ [J_{2n}(\varepsilon) J_{2k-2n}(\varepsilon) - \delta_{n,0} \delta_{k-n,0}] [g_{00}(\omega' + \Omega_{2n}) - g_{33}(\Omega_{2n})] + J_{2n-1}(\varepsilon) J_{2k-2n+1}(\varepsilon) [g_{00}(\Omega_{2n-1}) - g_{33}(\omega' + \Omega_{2n-1})] \right\}$  and  $\Pi_{\omega'}^{(2k-1)} = \sum_{n=-\infty}^{\infty} J_n(\varepsilon) J_{2k-1-n}(\varepsilon) [g_{30}(\Omega_n) + g_{30}(\omega' + \Omega_n)]$ . By substituting this quantity into the Dyson equation  $(D_{\omega}^{cq0})^{-1} D_{\omega, \omega'}^{cq} - \int d\omega_1 \Pi_{\omega, \omega_1} D_{\omega_1, \omega'}^{cq} = \delta(\omega - \omega')$ , we obtain

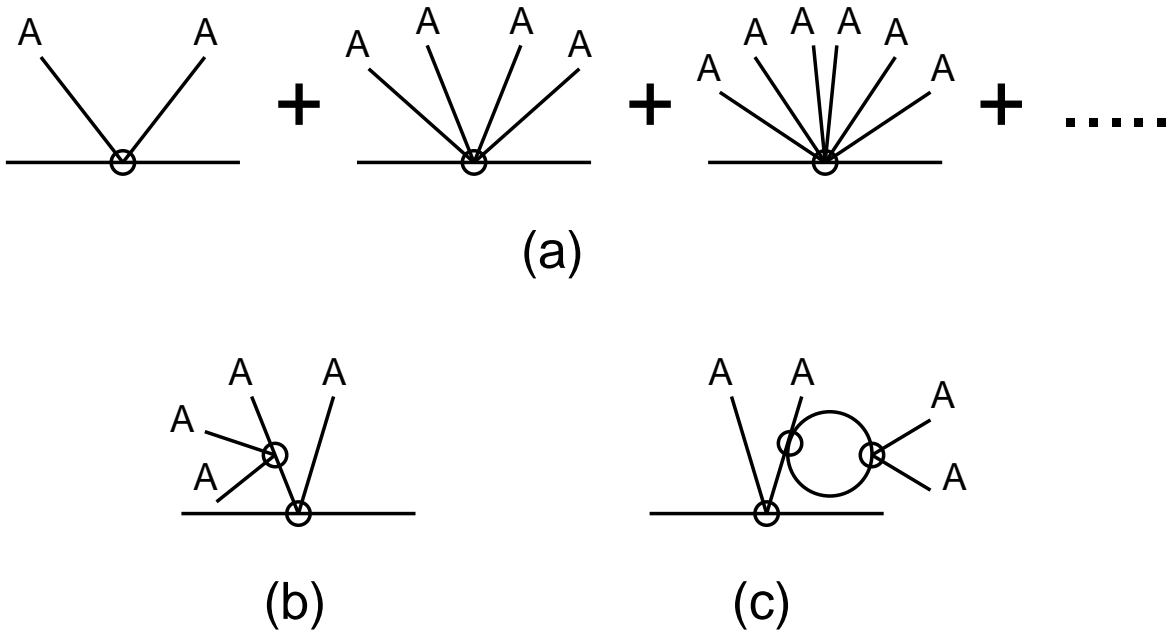
$$[(\omega + i0)^2 - \omega_0^2] D_{\omega, \omega'}^{cq} - \sum_{k=-\infty}^{\infty} \Pi_{\omega, \omega - k\Omega}^{(k)} D_{\omega - k\Omega, \omega'}^{cq} = 2E_C i \delta(\omega - \omega').$$

It can be shown that  $D_{\omega, \omega'}^{cq}$  represents an oscillating term in the case of  $\omega - \omega' = n\Omega$  by rewriting  $D_{\omega, \omega'}^{cq}$  as  $D_{\omega, \omega'}^{cq} = D_{(\omega+\omega')/2}(\omega - \omega') = \int dt D_{(\omega+\omega')/2}(t) e^{i(\omega-\omega')t}$ . This oscillating term can be shown to be small as compared with the case of  $\omega = \omega'$  in the dispersive region. Then, the solution of the Dyson equation is approximated by

$$D_{\omega, \omega'}^{cq} = \frac{2E_C i}{(\omega + i0)^2 - \omega_0^2 - \Pi_{\omega}^{(0)}} \delta(\omega - \omega') \quad (2)$$

(we put  $\Pi_{\omega, \omega}^{(0)} = \Pi_{\omega}^{(0)}$ ).

The diagram for  $\Pi_{\omega}^{(0)}$  is shown in Fig. 1(a). There are only even orders of  $A$  because the terms of the odd order do not contribute to  $\Pi_{\omega, \omega}^{(0)}$ . Figures. 1(b) and 1(c) are typical diagrams that are not included in  $\Pi_{\omega}^{(0)}$ . The former is a correction to Fig. 1(a). This



**Fig. 1.** (a) Types of diagrams that are numerically calculated in Sect. 3. (b) Example of other tree-type graphs. (c) Example of the loop correction.  $A$  indicates the external field. The solid line represents  $D^{cq0}$ ,  $D^{qc0}$ , or  $D^{cc0} = \coth \frac{\omega}{2T} (D^{cq0} - D^{qc0})$ . The small circle indicates the interaction vertex.

term can be obtained by replacing  $D^0$  with  $D$ , but it is small in the perturbative regime as the numerical calculation shows. The latter is the loop correction that is mentioned above. The contribution of this term to  $\Pi_\omega^{(0)}$  is also negligible in the dispersive region.

### 3. Damping Rate by Quasiparticle Excitations in External Fields

From Eq. (2), the damping rate of Josephson plasmons is given by  $\gamma_\omega = \frac{-1}{2\tilde{\omega}_0} \text{Im}\Pi_\omega^{(0)}$  in the case that  $|\text{Im}\Pi_\omega^{(0)}| \ll \tilde{\omega}_0^2$  ( $\tilde{\omega}_0$  is defined below). Here, we omit the imaginary part of  $\omega_0$ , which is proportional to  $\text{Im}E_J(\omega) \propto e^{-\Delta/T}$ ,<sup>16)</sup> and

$$-\text{Im}\Pi_\omega^{(0)} = \sum_{n=-\infty}^{\infty} \left\{ \delta_{n,0} \text{Im}g_{00}(\omega + \Omega_{2n}) - [J_{2n}(\varepsilon)]^2 \text{Im}g_{00}(\omega + \Omega_{2n}) - [J_{2n-1}(\varepsilon)]^2 \text{Im}g_{33}(\omega + \Omega_{2n-1}) \right\}.$$

This is written as follows at low temperatures ( $T \rightarrow 0$ ).

$$-\text{Im}\Pi_\omega^{(0)} = 4\pi N^2 E_C (\rho_0 t')^2 \Delta \sum_{n=-\infty}^{\infty} [J_n(\varepsilon)]^2 \int_1^{|w_n|-1} dx \frac{(|w_n| - x)x - (-1)^n \cos\varphi}{\sqrt{(|w_n| - x)^2 - 1} \sqrt{x^2 - 1}} \Big|_{|w_n|>2} \text{sgn}(w_n). \quad (3)$$

Here,  $w_n := (\omega + n\Omega)/\Delta$  and  $\varphi := \phi_L - \phi_R$ .<sup>17)</sup> [ $A|_B$  indicates that  $A|_B = A$  (0) if a condition  $B$  is satisfied (not satisfied).] In the same way, the Josephson plasma

frequency ( $\tilde{\omega}_0$ ) shifted by the effect of external fields is written as follows.

$$\begin{aligned}
\tilde{\omega}_0^2 &= \omega_0^2 + \text{Re}\Pi_\omega^{(0)} \\
&= \sum_{n=-\infty}^{\infty} \left\{ [J_{2n}(\varepsilon)]^2 [\text{Reg}_{00}(\omega + \Omega_{2n}) - \text{Reg}_{33}(\Omega_{2n})] - [J_{2n-1}(\varepsilon)]^2 [\text{Reg}_{00}(\Omega_{2n-1}) - \text{Reg}_{33}(\omega + \Omega_{2n-1})] \right\} \\
&= 4\pi N^2 E_C (\rho_0 t')^2 \Delta \sum_{n=-\infty}^{\infty} [J_n(\varepsilon)]^2 \left[ \int_1^{1+|w_n|} dx \frac{(x - |w_n|)x + (-1)^n \cos\varphi}{\sqrt{1 - (x - |w_n|)^2} \sqrt{x^2 - 1}} \Big|_{0 < |w_n| < 2} \right. \\
&\quad + \int_{-1}^1 dx \frac{(x + |w_n|)x + (-1)^n \cos\varphi}{\sqrt{(x + |w_n|)^2 - 1} \sqrt{1 - x^2}} \Big|_{|w_n| > 2} - \int_1^{1+|w'_n|} dx \frac{(x - |w'_n|)x - (-1)^n \cos\varphi}{\sqrt{1 - (x - |w'_n|)^2} \sqrt{x^2 - 1}} \Big|_{0 < |w'_n| < 2} \\
&\quad \left. - \int_{-1}^1 dx \frac{(x + |w'_n|)x - (-1)^n \cos\varphi}{\sqrt{(x + |w'_n|)^2 - 1} \sqrt{1 - x^2}} \Big|_{|w'_n| > 2} \right]
\end{aligned} \tag{4}$$

( $w'_n = n\Omega/\Delta$ ). As we consider the case of  $T = 0$ , these results do not show the effect of the thermal excitations but the photoassisted tunneling of quasiparticles, which is also observed in the current-voltage characteristic<sup>18)</sup> and the shot noise in the junctions of normal metals<sup>19)</sup> and normal-metal–superconductor.<sup>20)</sup>

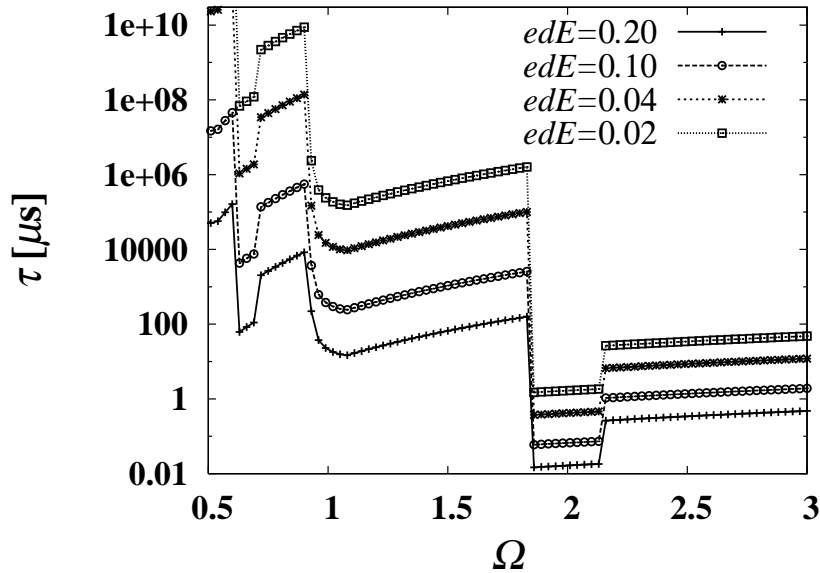
### 3.1 Numerical calculations

In numerical calculations, we take the amplitude of the superconducting gap as the unit of energy ( $\Delta = 1$ ). The relaxation time ( $\tau = \hbar/\gamma_\omega$ ) is quantitatively calculated by putting this unit of energy at  $2 \times 10^{-4}$  eV. To eliminate the common numerical factor  $\mathcal{N}_c = 4\pi N^2 E_C (\rho_0 t')^2 \Delta$ , we introduce the following quantities:  $P(\varphi) = -\text{Im}\Pi_\omega^{(0)}/\mathcal{N}_c$  and  $Q(\varphi) = \tilde{\omega}_0^2/\mathcal{N}_c$  from Eqs. (3) and (4), respectively. Then, the damping rate at  $\omega = \tilde{\omega}_0$  is written as

$$\gamma_\omega = \frac{-\text{Im}\Pi_\omega^{(0)}}{2\tilde{\omega}_0} = \frac{\tilde{\omega}_0|_{\varphi=0} P(\varphi)}{2\sqrt{Q(\varphi)Q(\varphi=0)}}. \tag{5}$$

The numerical calculations are performed by fixing the value of  $\tilde{\omega}_0|_{\varphi=0}$  ( $\tilde{\omega}_0 \neq \tilde{\omega}_0|_{\varphi=0}$  in the case of  $\varphi \neq 0$ ).  $\omega$  in  $P(\varphi)$  and  $Q(\varphi)$  is approximated as  $\omega = \tilde{\omega}_0 \simeq \omega_0$ , which is verified in the dispersive region, as discussed below.

The dependences of the relaxation time  $\tau$  on  $\Omega$  for several values of  $edE$  are shown in Fig. 2. Here, we put  $\tilde{\omega}_0|_{\varphi=0} = 0.15$  and  $\varphi = 0$ . There are several values of  $\Omega$  at which  $\tau$  shows a discontinuity. These points are found at  $\Omega = \frac{2\Delta+\omega}{n}$  and  $\frac{2\Delta-\omega}{n}$  ( $n$  is an integer).



**Fig. 2.** Dependences of the relaxation time  $\tau$  on  $\Omega$  for several values of  $edE$ .  $\tilde{\omega}_0|_{\varphi=0} = 0.15$  and  $\varphi = 0$ . The result is shown in microseconds by setting  $\Delta = 2 \times 10^{-4}$  eV.

This can be shown by rewriting  $P(\varphi)$  as  $\sum_{n=1}^{\infty} [J_n(\varepsilon)]^2 p_n(\varphi)$ . Here,

$$p_n(\varphi) = \int_1^{w_n-1} dx \frac{(w_n - x)x - (-1)^n \cos \varphi}{\sqrt{(w_n - x)^2 - 1} \sqrt{x^2 - 1}} - \int_1^{-w_{-n}-1} dx \frac{(-w_{-n} - x)x - (-1)^n \cos \varphi}{\sqrt{(-w_{-n} - x)^2 - 1} \sqrt{x^2 - 1}} \quad (6)$$

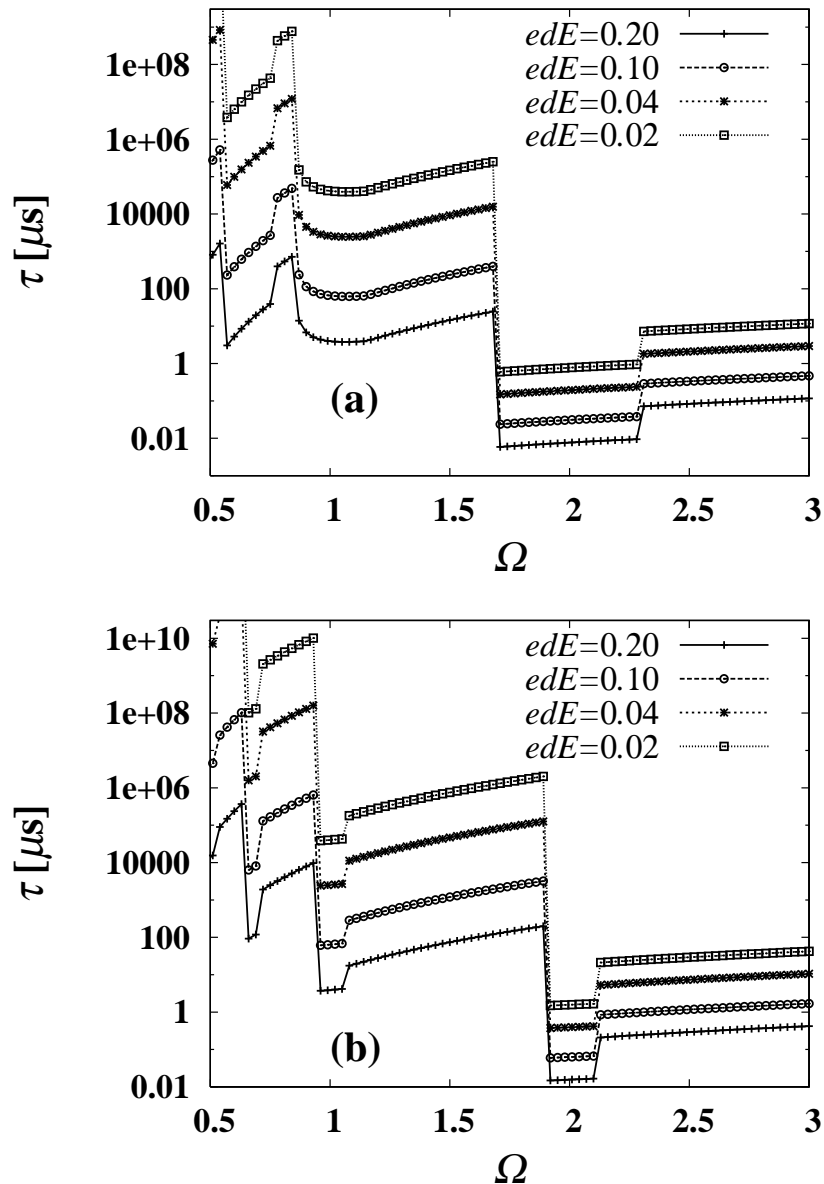
for  $\frac{2\Delta + \omega}{n} < \Omega < \frac{2\Delta - \omega}{n-1}$ , and

$$p_n(\varphi) = \int_1^{w_n-1} dx \frac{(w_n - x)x - (-1)^n \cos \varphi}{\sqrt{(w_n - x)^2 - 1} \sqrt{x^2 - 1}} \quad (7)$$

for  $\frac{2\Delta - \omega}{n} < \Omega < \frac{2\Delta + \omega}{n}$ . In our calculation, we consider the case of  $\varepsilon < 1$ , and then discontinuous steps arise at  $\Omega = (2\Delta - \omega)/n$ . On the other hand, the discontinuity at  $\Omega = (2\Delta + \omega)/n$  shows a step only if the graphs given by Eqs. (6) and (7) do not intersect with each other. If  $n$  is odd, this intersection does not exist and there is always a discontinuous step. In the case that  $n$  is even, the existence of the intersection depends on the value of  $\varphi$ , and the discontinuous step does not exist for  $\varphi \simeq 0$  as in Fig. 2.

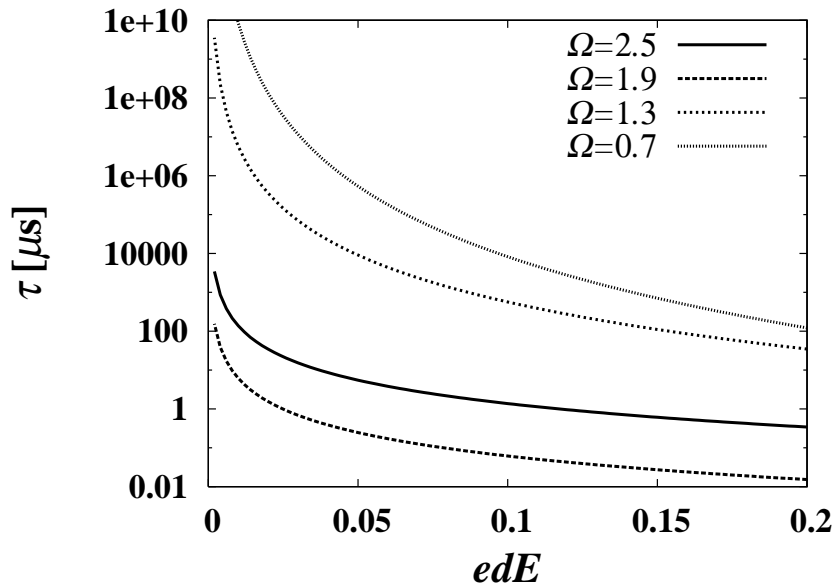
The dependences of  $\tau$  on  $\Omega$  for different values of  $\tilde{\omega}_0|_{\varphi=0}$  and  $\varphi$  are shown in Fig. 3. Both Figs. 3(a) and 3(b) show the shifts of discontinuous points at  $\Omega = (2\Delta - \omega)/n$  and  $(2\Delta + \omega)/n$  from those of Fig. 2 because of different  $\omega = \tilde{\omega}_0$ . In Fig. 3(b), another discontinuous step exists at  $(2\Delta + \omega)/(2n)$  because  $\varphi \neq 0$  as noted above. As shown in Fig. 3(a),  $\gamma_\omega$  is enhanced by increasing  $\tilde{\omega}_0|_{\varphi=0}$ , which can be seen from Eq. (5) and the fact that both  $P(\varphi)$  and  $Q(\varphi)$  do not show notable quantitative changes except the shifts





**Fig. 3.** Dependences of the relaxation time  $\tau$  on  $\Omega$  for several values of  $edE$ . (a)  $\tilde{\omega}_0|_{\varphi=0} = 0.3$  and  $\varphi = 0$ . (b)  $\tilde{\omega}_0|_{\varphi=0} = 0.15$  and  $\varphi = \pi/3$ . The results are shown in microseconds by setting  $\Delta = 2 \times 10^{-4}$  eV.

by varying  $\omega$ . Then, the relaxation time decreases with increasing  $\tilde{\omega}_0|_{\varphi=0}$ . On the other hand, the relaxation time does not show a noteworthy quantitative difference between Figs. 2 and 3(b) except the shift of steps. The reasons for this are that  $\tilde{\omega}_0|_{\varphi=0} = 0.15$  is common to both cases, and both  $P(\varphi)$  and  $Q(\varphi)$  in Eq. (5) are decreasing functions with decreasing  $\cos\varphi$ . These results indicate that the relaxation time by nonequilibrium quasiparticle excitations is sensitive to the intrinsic quantity  $\tilde{\omega}_0|_{\varphi=0}$  which is dependent



**Fig. 4.** Dependences of the relaxation time  $\tau$  on  $edE$  for several values of  $\Omega$ .  $\tilde{\omega}_0|_{\varphi=0} = 0.15$  and  $\varphi = 0$ . The result is shown in microseconds by setting  $\Delta = 2 \times 10^{-4}$  eV.

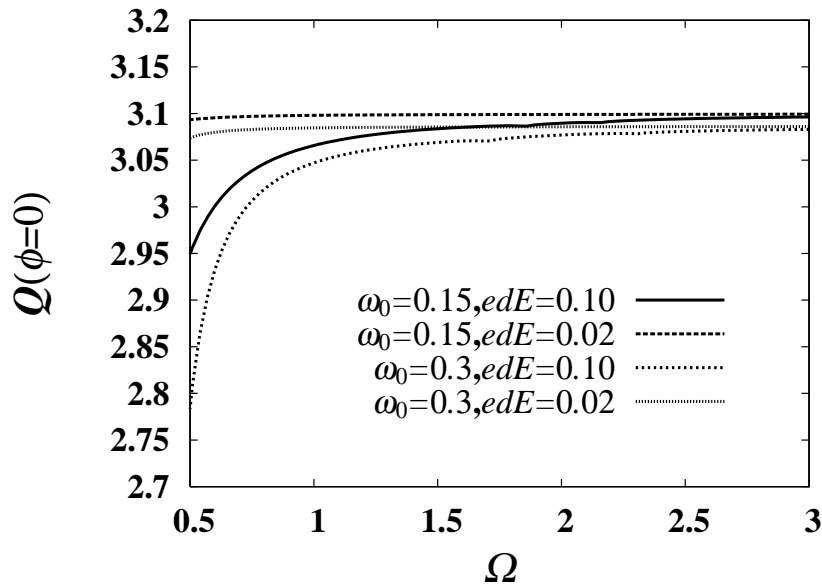
on the materials, and insensitive to the extrinsic parameter  $\varphi$ .

The dependences of  $\tau$  on  $edE$  for several values of  $\Omega$  are shown in Fig. 4.  $\Omega = 2.5$ , 1.9, 1.3, and 0.7 correspond to the regions  $\Omega > 2\Delta + \omega$ ,  $2\Delta - \omega < \Omega < 2\Delta + \omega$ ,  $(2\Delta + \omega)/2 < \Omega < 2\Delta - \omega$ , and  $(2\Delta - \omega)/3 < \Omega < (2\Delta + \omega)/3$ , respectively. As noted above, the dependence of  $\gamma_\omega$  on  $edE$  is expected to change by varying the value of  $\Omega$ , and the graphs for  $\Omega = 2.5$ , 1.9, 1.3, and 0.7 can be fitted to  $0.014/(edE)^2$ ,  $0.00062/(edE)^2$ ,  $0.055/(edE)^4$ , and  $0.008/(edE)^6$ , respectively. The quantitative difference between  $\Omega = 2.5$  and 1.9 arises from Eqs. (6) and (7). This result shows that even the external field with  $\Omega < 2\Delta$  leads to a short relaxation time.

As a comparison with the above results, we quantitatively estimate the effect of thermally excited quasiparticles on the damping rate, which is omitted above and is written in the same way as Eq. (5):  $\gamma_\omega^{th} = \frac{\tilde{\omega}_0|_{\varphi=0} P^{th}(\varphi)}{2\sqrt{Q(\varphi)Q(\varphi=0)}}$ . Here

$$P^{th}(\varphi) \simeq (1 - e^{-\omega/T}) \int_1^\infty dx \frac{2e^{-x\Delta/T} [(x + \omega/\Delta)x + \cos\varphi]}{\sqrt{(x + \omega/\Delta)^2 - 1} \sqrt{x^2 - 1}}.$$

A numerical calculation shows that the relaxation time  $\tau^{th} = \hbar/\gamma^{th}$  increases from 1.5  $\mu\text{s}$  at  $T = 0.1$  to  $3.6 \times 10^4 \mu\text{s}$  at  $T = 0.05$  (110  $\mu\text{s}$  at  $T = 0.07$ ) for  $\tilde{\omega}_0 = 0.15$  and  $\varphi = 0$ . Then, at low temperatures ( $T \ll \tilde{\omega}_0$ ), quasiparticles excited by external fields are predominant over thermally excited quasiparticles in the damping rate of Josephson



**Fig. 5.** Dependences of  $Q(\varphi = 0)$  ( $= \tilde{\omega}_0^2/\mathcal{N}_c$ ) on  $\Omega$  for several values of  $\tilde{\omega}_0|_{\varphi=0}$  and  $edE$ . This quantity is dimensionless.

plasmons.

The dependences of  $Q(\varphi = 0)$  on  $\Omega$  for several values of  $\tilde{\omega}_0|_{\varphi=0}$  and  $edE$  are shown in Fig. 5. The effect of external fields is prominent for large  $\varepsilon = edE\alpha/\Omega$  as expected in Eq. (4). This effect is also enhanced in the case of large  $\tilde{\omega}_0$ , which originates from the factor  $\alpha = \Omega^2/(\Omega^2 - \omega_0^2)$ . The quantitative calculation shows, however, that  $|\tilde{\omega}_0 - \omega_0|/\tilde{\omega}_0 < 0.1$  for the dispersive region. This result justifies our perturbative calculation, and we can put  $\tilde{\omega}_0 \simeq \omega_0$  in this region.

#### 4. Summary and Discussion

In this study, we consider the effect of external fields on the damping rate of Josephson plasmons. A formula for the damping rate in the nonresonant case is obtained by a time-dependent path integral method, and this quantity is numerically calculated. The result shows that the relaxation time discontinuously changes by varying the frequency of the external field. The quantitative estimation indicates that prominent decay occurs even in the case of  $\Omega < 2\Delta$ .

Although the nonequilibrium quasiparticle excitation has been considered as a possible cause of damping, this effect has not been quantitatively estimated. This is because the origin of the nonequilibrium quasiparticles is not specified,<sup>5)</sup> and the coupling to the environment is not derived from the tunneling term but introduced separately.<sup>21)</sup>

In this study, the microscopic derivation of the coupling between external fields and Josephson plasmons makes a quantitative estimation possible.

Here, we comment on the other effects of external fields. These are taken into account by writing the effective action as  $iS_{\text{eff}} = \frac{-1}{2} \int d\omega (\phi_{-\omega}^c \phi_{-\omega}^q) \hat{m}_{\omega}^{\phi} \begin{pmatrix} \phi_{\omega}^c \\ \phi_{\omega}^q \end{pmatrix} + \frac{-1}{2} \int d\omega \sum_q (\phi_{-\omega}^c \phi_{-\omega}^q) \hat{m}_{\omega,q}^{\phi A} \begin{pmatrix} A_{\omega,q}^c \\ A_{\omega,q}^q/2 \end{pmatrix} + \frac{-1}{2} \int d\omega \sum_q (A_{-\omega,-q}^c A_{-\omega,-q}^q/2) \hat{m}_{\omega,q}^{A\phi} \begin{pmatrix} \phi_{\omega}^c \\ \phi_{\omega}^q \end{pmatrix} + \frac{-1}{2} \int d\omega \sum_q (A_{-\omega,-q}^c A_{-\omega,-q}^q/2) \hat{m}_{\omega,q}^A \begin{pmatrix} A_{\omega,q}^c \\ A_{\omega,q}^q/2 \end{pmatrix}$ , for example. After integrating out  $A^{c,q}$ , the retarded Green's function for  $\phi$  is given by the element at the first row and the second column of the matrix:  $[\hat{m}_{\omega}^{\phi} - \sum_q \hat{m}_{\omega,q}^{\phi A} (\hat{m}_{\omega,q}^A)^{-1} \hat{m}_{\omega,q}^{A\phi}]^{-1}$ , which corresponds to  $D_{\omega}^{c,q}$ . In the case of spontaneous emission, the decay of Josephson plasmons is caused by the modes with  $\omega_q^A \leq \tilde{\omega}_0$ <sup>22)</sup> [ $\omega_q^A$  are poles of  $(\hat{m}_{\omega,q}^A)^{-1}$ ]. The damping rate can be calculated after specifying the  $q$ -dependences of  $\hat{m}_{\omega,q}$ , which are determined from details of the shapes and configurations of the junction and the cavity. In the case of the Purcell effect, the decay is caused by the damping rate of the external fields,  $\kappa$  (the imaginary part of  $\omega_q^A$ ), which has its origin in the absorption by the cavity and depends on the properties of the material. ( $\kappa$  arises also from the coupling between different modes.<sup>23)</sup>) Therefore, the above two effects have more extrinsic factors than quasiparticle excitations, and a further study is required in order to estimate the damping rate quantitatively.

Although experimentally it is difficult to distinguish between the Purcell effect and the effect of quasiparticle excitations, the exclusion of modes around  $2\Delta - \omega < \Omega < 2\Delta + \omega$  from the cavity would increase the lifetime of qubits if the latter is a predominant damping effect. In this study, we do not investigate the effect of the resonant mode ( $\Omega \simeq \tilde{\omega}_0$ ). The main role of this mode is to induce the Rabi oscillation, and it is improbable that this mode excites quasiparticles because of  $\Omega \ll \Delta$ , as shown in Sect. 3. A further study is needed to clarify whether there are other types of damping effect.

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