Damping Rate of Josephson Plasmons by Quasiparticle Excitation in Small Tunnel Junctions

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We investigate the intrinsic damping rate of Josephson plasmons in small tunnel junctions. The effect of interactions between collective modes is calculated on the basis of the perturbative expansion. This expansion is valid for small values of E_C/E_J (E_C and E_J are the charging and Josephson coupling energies, respectively). The rage of damping by quasiparticle excitation is dependent on this ratio and is qualitatively different from the lowest-order term studied in previous works. The relaxation time is roughly a quadratic function of frequency, which is peculiar to small tunnel junctions. A small value of E_C/E_J does not necessarily mean a long relaxation time owing to the nonmonotonic dependence of the damping rate on frequency. These nontrivial properties originate from the tunneling term, which depends on frequency and describes both the damping and interaction effects.

KEYWORDS: quasiparticle excitation, perturbative expansion, tunnel junction, superconducting quantum bit, Josephson plasmon, relaxation time

1. Introduction

The superconducting quantum bit (qubit) with use of a small tunnel junction is a possible constituent element for quantum information processing.[1] Although there are many other physical systems considered as candidates for quantum bits,[2] superconducting systems have an advantage in that existing techniques for semiconductor technologies may be available for their fabrication. This may make it possible for an integrated circuit with tunnel junctions to manipulate superconducting qubits.

One of the disadvantages of using superconducting qubits is that the relaxation time is short compared with that of some other physical systems.[2] There is a possibility that this originates from the significant influence of the surroundings. Several possible causes of this have been investigated so far (phonon radiation,[3] dielectric loss from two-

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level states,[4] coupling with electromagnetic modes,[5] and nonequilibrium quasiparticle excitation[6]). Several experiments have shown that the lifetime of superconducting qubits is still increasing.[7] This indicates that the extrinsic factors which shorten the lifetime of qubits have not been removed so far.

Investigating the dependences of the relaxation time on several parameters would be useful for clarifying the origins of the damping effect. In experiments, it has been observed that the relaxation time increases exponentially with decreasing temperature and then saturates at some temperatures (for the transmon qubit[8,9] and flux qubit[10]). In another study,[5] Houck *et al.* investigated the dependence of the relaxation time on frequency and suggested that the lifetime is affected by the multimode Purcell effect because of the quadratic dependence accompanied by asymmetry.

In this paper, we theoretically investigate th intrinsic damping effect. In superconductors, quasiparticle excitation is inevitable at finite temperatures. We consider how this effect influences the damping rate of Josephson plasmons (an out-of-phase collective mode in the tunnel junction). In previous works, the effect of quasiparticle excitation has been considered only in the lowest order.[11] We show that nonlinear terms in the perturbative expansion bring about a nontrivial damping effect.

In §2, we give a formalism for the perturbative expansion of the tunneling term with superconducting phase fluctuations across the junction. A difference between previous works and our calculation is noted. In §3, the expression for the damping rate is derived and numerically computed. It is shown that the parameter of the expansion is E_C/E_J = (charging energy)/(Josephson coupling energy). The perturbative expansion qualitatively changes the dependence of the damping rate on frequency. In §4, a comparison between small and large tunnel junctions is mentioned, and a relation of our theory to experimental results is discussed. In the Appendix, we estimate corrections to the results in §3 by higher-order terms of the perturbative expansion, and discuss the validity of our calculation. We set $\hbar = 1$ in this paper.

2. Perturbative Expansion

We start with a noninteracting effective action for Josephson plasmons at the interface between the superconductor and insulating barrier derived in §4 of ref. 12:

$$S_0 = \frac{1}{2}\sum_q \phi_q D_0(q)^{-1}\phi_{-q}. \tag{1}$$

Here, we put $\phi_q = \Phi^{-}_{q,1}$ [$=(\Phi^{\rm L}_q - \Phi^{\rm R}_q)/\sqrt{2}$; $\Phi^{L(R)}_q$ indicates the phase fluctuation in the left (right) superconductor] and omit the suffixes 1 and $-$, which indicate 'at the interface' and out of phase, respectively. We can put $q = (q_0, q_x, q_y) = (q_0, 0, 0)$ for small junctions. ($q_0 = \omega_l = 2\pi T l$ is the Matsubara frequency; T is the temperature and l is an integer). This is because finite values of q_x and q_y induce excitation at high energies, as discussed in §5 of ref. 12. With this approximation,

$$D_0(q)^{-1} = -\frac{(\mathrm{i}\omega_l)^2 - \Omega_\gamma^2}{2\epsilon_c}.$$

Here, $\Omega_\gamma = \sqrt{\epsilon_j \epsilon_c}$, $\epsilon_c = 2\mathrm{e}^2(1/\bar{v}_q + 1/\bar{Q}^{00})$, and $\epsilon_j = -Q^{\rm LR}/\mathrm{e}^2$ (ϵ_j is defined below, $\bar{Q}^{00}$ indicates the screening effect of the Coulomb interaction, and $\bar{v}_q = 1/2\pi d$ with the lattice constant $a = 1$ and the width of the junction d).

Next we take account of higher-order terms to investigate the interaction effect of ϕ_q. We consider the following term derived by integrating out electrons in the effective action (with referring to Appendix A of ref. 12):

$$-\mathrm{Trln}(-\hat{G}^{-1}+\hat{W}) = -\mathrm{Trln}(-\hat{G}^{-1}) + \frac{1}{2}\mathrm{Tr}[\hat{G}\hat{W}\hat{G}\hat{W}] + \frac{1}{4}\mathrm{Tr}[\hat{G}\hat{W}\hat{G}\hat{W}\hat{G}\hat{W}\hat{G}\hat{W}] + \cdots. \quad (2)$$

Here, we take account of only the tunneling term because other terms are irrelevant for the discussion of Josephson plasmons. $\hat{G} = \begin{pmatrix} \hat{G}^L_k & \hat{0} \\ \hat{0} & \hat{G}^R_k \end{pmatrix}$ is Green's function of electrons in superconductors,

$$\hat{W} = t' \sqrt{\frac{1}{\beta N^2}} \frac{2}{N_z + 1} \left[W^{(0)}_q \begin{pmatrix} \hat{0} & \hat{\tau}_3 \\ \hat{\tau}_3 & \hat{0} \end{pmatrix} + W^{(1)}_q \begin{pmatrix} \hat{0} & -\hat{1} \\ \hat{1} & \hat{0} \end{pmatrix} \right],$$

$$W^{(0)}_q = (\beta N^2)^{1/2}\delta_{q,0} + \left(\frac{1}{\beta N^2}\right)^{1/2} \sum_{q_1} \frac{1}{2} w_{q_1} w_{q-q_1} + \left(\frac{1}{\beta N^2}\right)^{3/2} \sum_{q_1,q_2,q_3} \frac{1}{4!} w_{q_1} w_{q_2} w_{q_3} w_{q-q_1-q_2-q_3} + \cdots,$$

$$W^{(1)}_q = w_q + \left(\frac{1}{\beta N^2}\right) \sum_{q_1,q_2} \frac{1}{3!} w_{q_1} w_{q_2} w_{q-q_1-q_2} + \cdots,$$

and $w_q = \mathrm{i}\phi_q/\sqrt{2}$ ($\beta = 1/T$, N^2 is the number of sites on the interface, and N_z is that in the perpendicular direction).

We consider a term proportional to t'^2 in eq. (2):

$$\begin{aligned}S_2 :=&\frac{1}{2}\mathrm{Tr}[\hat{G}\hat{W}\hat{G}\hat{W}]\\ =&\frac{t'^2}{4}\sum_q Q^{(2)}(q)\phi_q\phi_{-q} + \frac{-t'^2}{4!}\frac{1}{\beta N^2}\sum_{q_1,q_2,q_3} Q^{(4)}(q_1,q_2,q_3)\phi_{q_1}\phi_{q_2}\phi_{q_3}\phi_{-q_1-q_2-q_3}\\ &+\frac{2t'^2}{6!}\left(\frac{1}{\beta N^2}\right)^2\sum_{q_1,q_2,q_3,q_4,q_5} Q^{(6)}(q_1,q_2,q_3,q_4,q_5)\phi_{q_1}\phi_{q_2}\phi_{q_3}\phi_{q_4}\phi_{q_5}\phi_{-q_1-q_2-q_3-q_4-q_5}+\cdots .\end{aligned} \tag{3}$$

Here, $Q^{(2)}(q) = Q_1(q) - Q_3(0) =: \epsilon_j/t'^2$, $Q^{(4)}(q_1,q_2,q_3) = [4Q_1(q_3) - Q_3(0) - 3Q_3(q_1 + q_3)]/4$, and $Q^{(6)}(q_1,q_2,q_3,q_4,q_5) = [6Q_1(q_3)+10Q_1(q_1+q_2+q_3)-Q_3(0)-15Q_3(q_1+q_2)]/16$ with $Q_3(q) = \frac{1}{\beta N^2}\sum_k\left(\frac{2}{N_z+1}\right)^2\sum_{\zeta_1,\zeta_2}\mathrm{Tr}[\hat{G}^L_{k+q,\zeta_1}\hat{\tau}_3\hat{G}^R_{k,\zeta_2}\hat{\tau}_3+\hat{G}^R_{k+q,\zeta_1}\hat{\tau}_3\hat{G}^L_{k,\zeta_2}\hat{\tau}_3]$ and $Q_1(q) = \frac{1}{\beta N^2}\sum_k\left(\frac{2}{N_z+1}\right)^2\sum_{\zeta_1,\zeta_2}\mathrm{Tr}[\hat{G}^L_{k+q,\zeta_1}\hat{G}^R_{k,\zeta_2}+\hat{G}^R_{k+q,\zeta_1}\hat{G}^L_{k,\zeta_2}]$. Then the term that describes the interaction effects is written as $S_I = S_2 - \sum_q \epsilon_j\phi_q\phi_{-q}/4$ [the second term on the right-hand side is already included in eq. (1) and excluded from S_I], and the effective action is written as $S_{\mathrm{eff}} = S_0 + S_I$. We calculate the damping rate on the basis of this action. Hereafter, we assume that the two superconductors of the Josephson junction are the same, namely, $\Delta_{L,R} = \Delta^0_{L,R}\mathrm{e}^{\mathrm{i}\varphi_{L,R}} = \Delta$; $\Delta^0_{L,R}$ and $\varphi_{L,R}$ are the amplitude and static phase of the superconducting order parameter, respectively.

Here, we mention the relationship between previous theories and our formulation. If we neglect the dependence of $Q^{(2)}$,$Q^{(4)}$,$Q^{(6)}$,... on $q = (q_0, \boldsymbol{q})$ and ϕ_q on $\boldsymbol{q}$, then

$$\begin{aligned}S_0 + S_I &= \int_0^\beta \mathrm{d}\tau\left[\frac{1}{8E_C}\left(\frac{\partial\tilde{\phi}_\tau}{\partial\tau}\right)^2 + \frac{E_J}{4}\left(\frac{1}{2}\tilde{\phi}^2_\tau - \frac{1}{4!}\tilde{\phi}^4_\tau + \frac{1}{6!}\tilde{\phi}^6_\tau + \cdots\right)\right]\\ &= \int_0^\beta \mathrm{d}\tau\left[\frac{1}{8E_C}\left(\frac{\partial\tilde{\phi}_\tau}{\partial\tau}\right)^2 + \frac{E_J}{4}(1-\cos\tilde{\phi}_\tau)\right].\end{aligned}$$

Here, we use the relations $\phi_q = \sqrt{\frac{N^2}{2\beta}}\int_0^\beta \mathrm{d}\tau \mathrm{e}^{\mathrm{i}\omega_l\tau}\tilde{\phi}_\tau$, $\epsilon_j = E_J/N^2$, and $\epsilon_c = N^2E_C$. This action is equivalent to the Hamiltonian of the small tunnel junction, which is used to describe the competition between the phase and number fluctuation of the Cooper pair.[13] We show below, however, that the dependences of $Q^{(n)}$ on q, which are neglected in the above simplification, have nontrivial effects on the damping term.

3. Damping Rate of Josephson Plasmons

3.1 Imaginary part of the self-energy

The interaction effect is obtained by calculating the self-energy term $\Pi(q)$ in the propagator:

$$\begin{aligned} D(q) &= [D_0(q)^{-1} - \Pi(q)]^{-1} \\ &= -2\epsilon_c \left[(\mathrm{i}\omega_l)^2 - \Omega_\gamma^2 + 2\epsilon_c \Pi(q)\right]^{-1}. \end{aligned}$$

We decompose $\Pi(q)$ by the order of S_I as $\Pi(q) = \sum_i \Pi_i(q)$. Then $\Pi_i(q)$ are given by $\Pi_1(q) = 2\frac{\delta \ln Z_1}{\delta D_0(q)}$, $\Pi_2(q) = 2\frac{\delta \ln Z_2}{\delta D_0(q)} - \Pi_1(q) D_0(q) \Pi_1(q)$, ... with $\ln Z_1 = \int \mathcal{D}\phi \mathrm{e}^{-S_0}(-S_I)/Z_0$, $\ln Z_2 = [\int \mathcal{D}\phi \mathrm{e}^{-S_0} S_I^2/Z_0 - (\ln Z_1)^2]/2$, ... ($Z_0 = \int \mathcal{D}\phi \mathrm{e}^{-S_0}$). (This derivation follows the method described, for example, in ref. 14.)

We consider the case of $S_I = \frac{-t'^2}{4!}\frac{1}{\beta N^2}\sum_{q_1,q_2,q_3} Q^{(4)}(q_1,q_2,q_3)\phi_{q_1}\phi_{q_2}\phi_{q_3}\phi_{-q_1-q_2-q_3}$ by neglecting $Q^{(n)}$ $(n \geq 6)$ in eq. (3), and calculate the self-energy within the second order of this S_I: $\Pi(q) = \Pi_1(q) + \Pi_2(q)$. With the use of $\epsilon_j(q)/t'^2 = Q_1(q) - Q_3(0)$,

$$\Pi_1(q) = \frac{1}{2\beta N^2}\sum_{q_1}\frac{1}{2}[\epsilon_j(q) + \epsilon_j(q_1)] D_0(q_1) \tag{4}$$

and

$$\begin{aligned} \Pi_2(q) =& \frac{1}{6}\left(\frac{1}{\beta N^2}\right)^2 \sum_{q_1,q_2}\frac{1}{4^2}\left[\epsilon_j(q) + \epsilon_j(q_1) + \epsilon_j(q_2) + \epsilon_j(q+q_1+q_2)\right]^2 D_0(q_1)D_0(q_2)D_0(-q-q_1-q_2) \\ &+ \left(\frac{1}{\beta N^2}\right)^2 \sum_{q_1,q_2}\frac{1}{4^2}\left[\epsilon_j(q) + \epsilon_j(q_2)\right]\left[\epsilon_j(q_1) + \epsilon_j(q_2)\right] D_0(q_2)D_0(-q_2)D_0(q_1). \end{aligned} \tag{5}$$

Here, we neglect the q dependence of $Q_3(q)$. This approximation is supported by a numerical calculation which shows that $\mathrm{Im}Q_3^R(\omega)$ is smaller than $\mathrm{Im}Q_1^R(\omega)$ by one order of magnitude.

The damping rate in the noninteracting case (without the effect of S_I) is written as

$$\gamma_\omega = -\frac{\epsilon_c}{2\Omega}\mathrm{Im}\epsilon_j^R(\omega) = -\frac{\epsilon_c t'^2}{2\Omega}\mathrm{Im}Q_1^R(\omega) = \frac{-\Omega \mathrm{Im}Q_1^R(\omega)}{2\mathrm{Re}[Q_1^R(\omega) - Q_3^R(0)]}. \tag{6}$$

Here $\Omega_\gamma^2 = \epsilon_c \mathrm{Re}\epsilon_j^R(\omega) + \mathrm{i}\epsilon_c \mathrm{Im}\epsilon_j^R(\omega)$ and $\Omega^2 = \epsilon_c \mathrm{Re}\epsilon_j^R(\omega)$ (the suffix R means 'retarded', *i.e.*, the analytic continuation $\mathrm{i}\omega_l \to \omega + \mathrm{i}0$). We investigate how this term is modified by taking account of the interaction term S_I. This is written as

$$\tilde{\gamma}_\omega = -\frac{\epsilon_c}{2\Omega}\mathrm{Im}[\epsilon_j^R(\omega) - 2\Pi^R(\omega)] = \gamma_\omega\left[1 + \frac{\epsilon_c \mathrm{Im}\Pi^R(\omega)}{\Omega\gamma_\omega}\right] \tag{7}$$

by neglecting the shift of Ω (the real part of $\Pi^R(\omega)$). The contribution from the firstorder term [eq. (4)] is

$$\frac{\epsilon_c \mathrm{Im}\Pi_1^R(\omega)}{\Omega\gamma_\omega} = -\frac{E_C}{2\Omega}\coth\frac{\Omega}{2T}. \tag{8}$$

The second-order term [eq. (5)] is decomposed into four parts: $\mathrm{Im}\Pi_2^R(\omega) = \mathrm{Im}\Pi_{2a}^R(\omega) + \mathrm{Im}\Pi_{2b}^R(\omega) + \mathrm{Im}\Pi_{2c}^R(\omega) + \mathrm{Im}\Pi_{2d}^R(\omega)$. Here, $\mathrm{Im}\Pi_{2a}^R(\omega) = \frac{\epsilon_j^2}{6N^4}\mathrm{Im}[T^2\sum_{q_1,q_2}\tilde{D}_{q1,q2}(q)]$, $\mathrm{Im}\Pi_{2b}^R(\omega) = \frac{\epsilon_j}{6N^4}\mathrm{Im}[\frac{3T^2}{2}\sum_{q_1,q_2}\epsilon_j(q_1)\tilde{D}_{q1,q2}(q)]$, $\mathrm{Im}\Pi_{2c}^R(\omega) = \frac{\epsilon_j\mathrm{Im}\epsilon_j^R(\omega)}{6N^4}\mathrm{Re}[\frac{T^2}{2}\sum_{q_1,q_2}\tilde{D}_{q1,q2}(q)]$, and $\mathrm{Im}\Pi_{2d}^R(\omega) = \frac{\epsilon_j\mathrm{Im}\epsilon_j^R(\omega)}{8N^4}T^2\sum_{q_1,q_2}D_0(q_2)D_0(-q_2)D_0(q_1)$. [The dependence of $\mathrm{Re}\epsilon_j^R(\omega)$ on ω is negligible compared with that of the imaginary part; and therefore, we write $\mathrm{Re}\epsilon_j^R(\omega)$ as ϵ_j. $\tilde{D}_{q1,q2}(q) := D_0(q_1)D_0(q_2)D_0(-q-q_1-q_2)$. Im[...] and Re[...] mean taking the imaginary and real parts after performing the integration with analytic continuation in [...], respectively.]

Hereafter, we assume that the system is at low temperatures: $T \ll \Delta$. For the range of frequencies in which γ can be omitted in the denominator ($i.e.$, $|\omega - n\Omega| \gg \gamma_{\omega-(n-1)\Omega}$ with $n = \pm1, \pm3$),[15]

$$\frac{\epsilon_c \mathrm{Im}\Pi_{2a}^R(\omega)}{\Omega\gamma_\omega} = \frac{E_C^2}{12(\sinh\Omega/2T)^2}\left[\frac{1+g_{-2}/2}{(\omega-\Omega)^2} + \frac{g_{-2}/2}{(\omega-3\Omega)^2} + \frac{1+g_{+2}/2}{(\omega+\Omega)^2} + \frac{g_{+2}/2}{(\omega+3\Omega)^2}\right], \tag{9}$$

$$\frac{\epsilon_c \mathrm{Im}\Pi_{2b}^R(\omega)}{\Omega\gamma_\omega} = \frac{E_C^2}{2(\sinh\Omega/2T)^2}\left[\frac{1}{\omega^2-\Omega^2} + \frac{g_{-2}/2}{(\omega-2\Omega)^2-\Omega^2} + \frac{g_{+2}/2}{(\omega+2\Omega)^2-\Omega^2}\right], \tag{10}$$

$$\begin{aligned}\frac{\epsilon_c \mathrm{Im}\Pi_{2c}^R(\omega)}{\Omega\gamma_\omega} =& \frac{E_C^2(\coth\Omega/2T)^2}{3}\Bigg\{\left[1+\frac{1}{2(\cosh\Omega/2T)^2}\right]\frac{1}{\omega^2-\Omega^2} \\ &+ \frac{1}{2}\left(1+\frac{\coth\Omega/T}{\coth\Omega/2T}\right)\left[\frac{1}{(\omega-2\Omega)^2-\Omega^2} + \frac{1}{(\omega+2\Omega)^2-\Omega^2}\right]\Bigg\},\end{aligned} \tag{11}$$

and

$$\frac{\epsilon_c \mathrm{Im}\Pi_{2d}^R(\omega)}{\Omega\gamma_\omega} = \frac{-E_C^2}{4\Omega^2}\left(\coth\frac{\Omega}{2T}\right)^2\left[1+\frac{\Omega/T}{\sinh(\Omega/T)}\right]. \tag{12}$$

Here, $g_{\pm n} = \gamma_{\omega\pm n\Omega}\sinh(\omega/2T)/[\gamma_\omega\sinh(\omega \pm n\Omega)/2T]$.

The above results show that the first- and second-order terms are proportional to $E_C/\Omega = \sqrt{E_C/E_J}$ and $(E_C/\Omega)^2 = E_C/E_J$, respectively. This indicates that the expansion is valid for small E_C/E_J such as the transmon qubit.[5,8,16] Higher-order terms other than the above terms are discussed in the Appendix.

In the case of $|\omega - \Omega| \simeq \gamma_\omega$, although this range is very narrow at low temperatures, a self-consistent calculation is required. From eq. (7), the solution is written as

$$\tilde{\gamma}_\Omega \simeq \frac{E_C}{2\sqrt{2}\sinh(\Omega/2T)}$$

with the second-order approximation as above. Then, at low temperatures, $\tilde{\gamma}_\Omega$ is proportional to $\exp(-\Omega/2T)$ and decreases slowly as compared with $\gamma_\Omega \propto \exp(-\Delta/T)$. The relaxation time at $\omega = \Omega$ is very short as compared with the results in the range of $|\omega - \Omega| \gg \gamma_\omega$, as shown in §3.2; for example, $1/\tilde{\gamma}_\Omega \simeq 8.68 \times 10^{-4}$ μs in the case of $E_C = 0.01$ and $T = 0.06$ (even in this case, the peak in the imaginary part of the propagator around $\omega \simeq \Omega$ is well defined because $\tilde{\gamma}_\Omega \simeq 0.0038$ in the unit of $\Delta = 1$).

3.2 Numerical calculation

We take the superconducting gap as the unit of energy ($\Delta = 1$) and fix Ω to $\Omega/\Delta = 0.10$. The relation between our definitions of E_C and E_J and those of previous works[16] (E'_C and E'_J) is given by $E_C = 2E'_C$ and $E_J = 4E'_J$. Then $\sqrt{E_C E_J} = \sqrt{8E'_C E'_J}$, and $\Omega/E_C = \sqrt{2E'_J/E'_C} = 10$ for $E'_J/E'_C = 50$ in a typical case of the transmon qubit. The results shown below are calculated in the range of $|\omega - \Omega| \gg \gamma_\omega$,[17] as noted in the previous subsection.

As shown in §3.1, the finite values of $\mathrm{Im}\Pi^R(\omega)$ originate from the existence of γ_ω in $D_0(q)$. The rate of damping by quasiparticle excitation without the interaction of Josephson plasmons is written as follows from eq. (6):

$$\gamma_\omega = \frac{\Omega \mathrm{sgn}(\omega)}{\Delta \tanh(\Delta/2T)} \Bigg[\int_\Delta^\infty \frac{\mathrm{d}\epsilon}{2\pi} \frac{2\mathrm{e}^{-\epsilon/T}(1-\mathrm{e}^{-|\omega|/T})}{[1+\mathrm{e}^{-(\epsilon+|\omega|)/T}](1+\mathrm{e}^{-\epsilon/T})} \frac{(\epsilon+|\omega|)\epsilon + \Delta^2}{\sqrt{(\epsilon+|\omega|)^2-\Delta^2}\sqrt{\epsilon^2-\Delta^2}}$$
$$+ \frac{\theta(|\omega|-2\Delta)}{2} \int_\Delta^{|\omega|-\Delta} \frac{\mathrm{d}\epsilon}{2\pi} \left(\tanh\frac{\epsilon}{2T} + \tanh\frac{|\omega|-\epsilon}{2T} \right) \frac{(|\omega|-\epsilon)\epsilon - \Delta^2}{\sqrt{(|\omega|-\epsilon)^2-\Delta^2}\sqrt{\epsilon^2-\Delta^2}} \Bigg],$$

with the approximation $\mathrm{Re}[Q_1(q) - Q_3(0)] \simeq Q_1(0) - Q_3(0)$. The second term is important only for the excitation across the superconducting gap and is omitted below for our discussion around $\omega \sim \Omega$. The dependences of $1/\gamma_\omega$ on ω for several temperatures are shown in Fig. 1. The calculated result for $1/\gamma_\omega$ is shown in μs by setting $\Delta = 2 \times 10^{-4}$ eV. $1/\gamma_\omega$ is proportional to $\exp(\Delta/T)$ as expected from the above expression.

The interaction between Josephson plasmons changes the above behavior (we consider the effect of $\mathrm{Im}\Pi$; the real part only gives the shift of Ω and is omitted). The dependences of $1/\tilde{\gamma}_\omega$ [eq. (7)] on ω for several values of E_C are shown in Fig. 2. For small $|\omega - \Omega|$, $1/\tilde{\gamma}_\omega$ is roughly proportional to $(\omega - \Omega)^2$; however, asymmetry around $\omega \approx \Omega$ exists. This asymmetric behavior becomes prominent with increasing E_C and T,

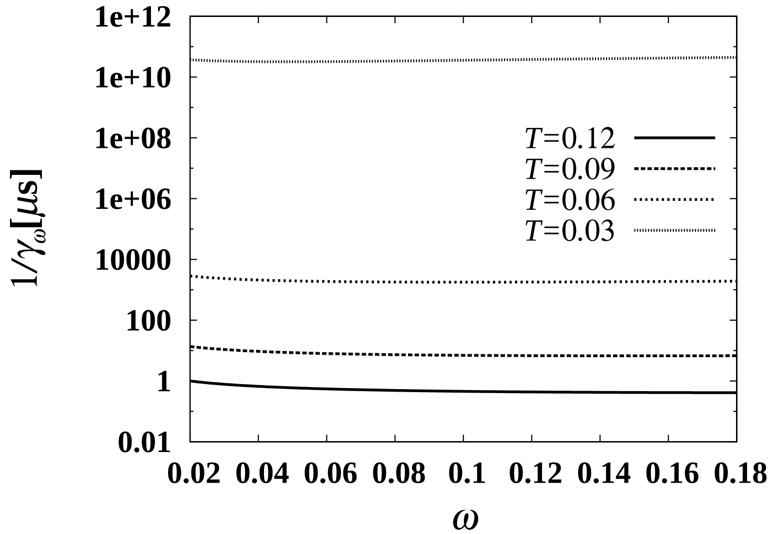

Fig. 1. Dependences of $1/\gamma_\omega$ on ω and T. This quantity is shown in microseconds with $\Delta = 2 \times 10^{-4}$ eV.

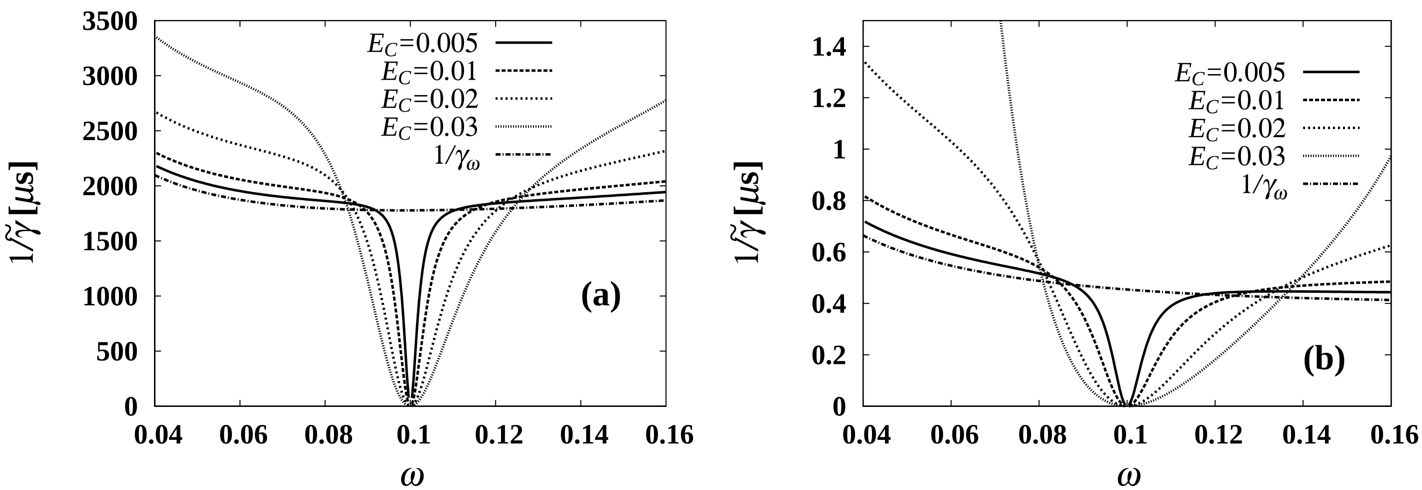

Fig. 2. Dependences of $1/\tilde{\gamma}_\omega$ on ω and E_C for (a) $T = 0.06$ and (b) $T = 0.12$. $1/\gamma_\omega$ is also shown for comparison. This quantity is shown in microseconds with $\Delta = 2 \times 10^{-4}$ eV. The graphs plotted are restricted to the range in which $|\omega - \Omega| \geq 10\gamma_\omega$ is satisfied.

which is easily seen if we graph $\gamma_\omega/\tilde{\gamma}_\omega$ as in Fig. 3. This behavior can be understood from the expressions for $\mathrm{Im}\Pi^R$ in §3.1. Equations (9)-(11) show significant frequency dependences although these terms vanish with decreasing temperature. On the other hand, eqs. (8) and (12) retain constant values. This explains the temperature dependences. Equation (9) provides proportionality to $(\omega - \Omega)^2$, and this results from the fact that Josephson plasmons contain a damping term caused by quasiparticle exci-

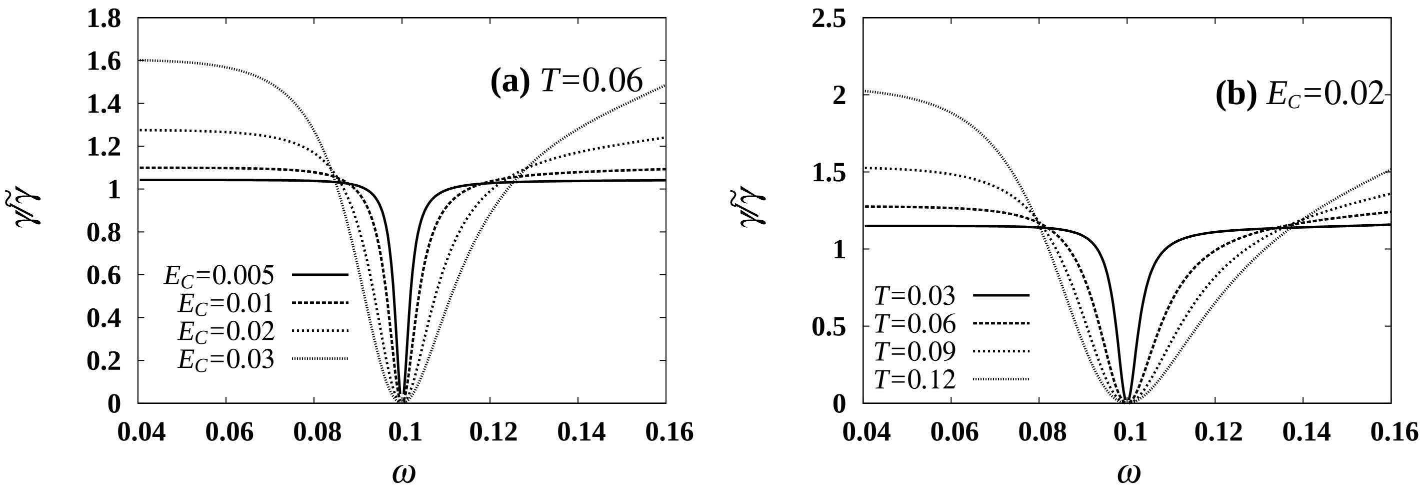

Fig. 3. Dependences of $\gamma_\omega/\tilde{\gamma}_\omega$ on ω for several values of E_C and T for (a) $T = 0.06$, (b) $E_C = 0.02$. The graphs plotted are restricted to the range in which $|\omega - \Omega| \geq 10\gamma_\omega$ is satisfied.

tation, namely, γ_ω in $D_0^R(\omega)$ (i). Equations (10) and (11) give negative and positive values for $\omega < \Omega$ and $\omega > \Omega$, respectively. This causes the asymmetric behavior around $\omega \approx \Omega$. This term has its origin mainly in the internal structure of the interaction vertex between Josephson plasmons, namely, the frequency dependence of ϵ_j (ii). These two effects [(i) and (ii)] are not included in the conventional ϕ^4 theory and are peculiar to systems in which the bosonic degree of freedom is constructed from fermionic degrees of freedom.

4. Summary and Discussion

In this paper, we calculated the damping rate of Josephson plasmons by quasiparticle excitation on the basis of the perturbation expansion. We investigated the properties inherent in small tunnel junctions by calculating the dependences of the damping rate on frequency and temperature without extrinsic effects. The damping rate is found to be roughly proportional to $1/(\omega - \Omega)^2$ around $\omega \simeq \Omega$ and asymmetric for $\omega < \Omega$ and $\omega > \Omega$. These are caused by the interaction between Josephson plasmons, as is confirmed by varying E_C. Previous works did not take account of interaction effects and considered only γ_ω as the effect of quasiparticle excitation.

The quadratic dependence of the relaxation time on frequency is peculiar to the case of small junctions. The narrow dimension of the junction restricts low-energy excitation accompanied by spatial variation. ($\Omega_{\boldsymbol{q}}$ is large when $\boldsymbol{q} \neq \mathbf{0}$ as estimated in ref. 12.) On the other hand, for a large junction, the dependence of $\Omega_{\boldsymbol{q}}$ (Josephson plasma frequency) on $\boldsymbol{q}$ (wave number) is important and the summation $\frac{1}{N^2}\sum_{\boldsymbol{q}}$ is replaced by the integral $\int \frac{\mathrm{d}^2 q}{(2\pi)^2}$. In this case, the damping rate in eq. (7) from the second order [eq. (5)] is written as

$$2\epsilon_c \mathrm{Im}\Pi_2^R(\boldsymbol{q},\omega) = \frac{\pi\epsilon_c^2\Omega\sinh(\omega/2T)}{12}\iint \frac{\mathrm{d}^2q_1\mathrm{d}^2q_2}{(2\pi)^4}\frac{\sum_{s1,s2,s3=\pm1}\delta(\omega+s_1\Omega_{q1}+s_2\Omega_{q2}+s_3\Omega_{q-q1-q2})}{\sinh(\Omega_{q1}/2T)\sinh(\Omega_{q2}/2T)\sinh(\Omega_{q-q1-q2}/2T)}.$$

From this equation, we can summarize the differences between small and large tunnel junctions as follows: The $1/(\omega-\Omega)^2$ behavior of the damping rate is smoothed away owing to the integration. The damping rate is proportional to $\exp(-\Omega/T)$ in contrast to $\tilde{\gamma}_\omega \propto \exp(-\Delta/T)$ in §3.

In experiments, exponential decay was observed in a restricted range of temperatures,[8,9] and it is quantitatively consistent with our calculation in the corresponding range of temperatures. The relaxation time reaches a constant value with decreasing temperature. This may originate from extrinsic effects that are not included in our theory. The observation of ω dependences may be helpful in clarifying the origin of this behavior. For example, if the relaxation is caused by nonequilibrium quasiparticles, as suggested in ref. 8, the dependence of the relaxation time on frequency will be the same as those of the calculated results in this paper.

Characteristic ω dependences similar to our calculated results were observed in experiments, and this behavior has been attributed to the multimode Purcell effect.[5] Our calculation shows, however, that the damping effect by quasiparticle excitation gives rise to the same ω dependences in the relaxation time. By measuring the dependence of the relaxation time on temperature, it should be determined whether the observed ω dependence originates from the intrinsic effect by quasiparticle excitation or extrinsic (Purcell) effects.

The transmon qubit has been said to have a long coherence time because of its insensitivity to charge noise.[7,16] This is based on the small values of E_C/E_J and is considered to be true as long as compared with the Cooper pair box ($E_C/E_J \approx 1$). Our calculation shows, however, that the rate of damping by quasiparticle excitation is dependent on E_J/E_C. This dependence varies with the frequency. The relaxation time is long for small values of E_C/E_J at a frequency close to Ω. On the other hand, this tendency is reversed with increasing detuning ($|\omega-\Omega|$). This behavior may be used to judge whether the observed damping rate originates from the intrinsic effect.

For further comparison with experiments, it is required to include the interaction with external fields because the manipulation of qubits cannot be carried out without this. In this paper, we fixed the frequency of Josephson plasmons (Ω) and varied the probing frequency (ω). In experiments, however, these roles are reversed; the cavity frequency is fixed and the qubit frequency is varied. The latter variation is implemented by applying a flux to the junction and making the difference between the static phases finite ($\varphi_L \neq \varphi_R$). Therefore, a theory including the difference between phases would reproduce a more precise experimental situation.

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Appendix: Higher-order terms

In this appendix, we consider the higher-order terms omitted in §3. These are classified into the following three types: (a) the terms of ϕ^n with $n \geq 6$ in eq. (3), (b) the self-energies Π_n with $n \geq 3$ in §3.1, and (c) the terms $\frac{1}{4}\mathrm{Tr}[\hat{G}\hat{W}\hat{G}\hat{W}\hat{G}\hat{W}\hat{G}\hat{W}] + \cdots$ in eq. (2).

Firstly, we consider the case of (a). If we take account of the third term in eq. (3) and write it as

$$S_I^{(6)} := \frac{2}{6!}\left(\frac{1}{\beta N^2}\right)^2 \sum_{q_1,q_2,q_3,q_4,q_5} \left[\frac{3}{8}\epsilon_j(q_3) + \frac{5}{8}\epsilon_j(q_1+q_2+q_3)\right] \phi_{q_1}\phi_{q_2}\phi_{q_3}\phi_{q_4}\phi_{q_5}\phi_{-q_1-q_2-q_3-q_4-q_5}$$

with the same approximation as in §3.1, then the calculation of Π_1 can be performed as above and the result is

$$\frac{\epsilon_c \mathrm{Im}\Pi_1^{(6)R}(\omega)}{\Omega\gamma_\omega} = \frac{E_C^2}{16\Omega^2}\left[\frac{2}{(\tanh\Omega/2T)^2} + \frac{1+(g_{+2}+g_{-2})/2}{(\sinh\Omega/2T)^2}\right].$$

This term is a correction to eq. (4).

For the calculation of Π_2, we use the approximation $\epsilon_j(q) \simeq \epsilon_j(0)$, which is valid in order to obtain the most divergent term, as discussed in §3. Then we put $S_I = -\frac{\epsilon_j}{4!}\sum_{q1,q2,q3}\phi_{q1}\phi_{q2}\phi_{q3}\phi_{-q1-q2-q3} + \frac{2\epsilon_j}{6!}\sum_{q1,q2,q3,q4,q5}\phi_{q1}\phi_{q2}\phi_{q3}\phi_{q4}\phi_{q5}\phi_{-q1-q2-q3-q4-q5}$, and the result is written as follows:

$$\begin{aligned}\frac{\epsilon_c \mathrm{Im}\Pi_2^R(\omega)}{\Omega\gamma_\omega} =& \frac{E_C^2}{12(\sinh\Omega/2T)^2}\left(1 - \frac{E_C}{\Omega}\coth\frac{\Omega}{2T}\right)^2 \left[\frac{1+g_{-2}/2}{(\omega-\Omega)^2} + \frac{g_{-2}/2}{(\omega-3\Omega)^2} + \frac{1+g_{+2}/2}{(\omega+\Omega)^2} + \frac{g_{+2}/2}{(\omega+3\Omega)^2}\right] \\ &+ \frac{E_C^4}{80\Omega^2(\sinh\Omega/2T)^4}\sum_{s=\pm1}\left[\frac{1+2g_{s2}/3}{(\omega+s\Omega)^2} + \frac{2g_{s2}/3+g_{s4}/6}{(\omega+3s\Omega)^2} + \frac{g_{s4}/6}{(\omega+5s\Omega)^2}\right].\end{aligned}$$

These results for Π indicate that the predominant term does not change from $1/(\omega-\Omega)^2$ and its coefficient is corrected by powers of E_C/Ω. Therefore, the results in §3 do not change qualitatively and are quantitatively valid in the case of transmon qubits ($E_C \ll E_J$).

Secondly, we consider the case of (b). The same argument as above is also applied to the higher-order terms given by

$$\Pi_n(q) \approx 2\frac{\delta}{\delta D_0(q)}\left[\frac{1}{n!}\frac{\int \mathcal{D}\phi \mathrm{e}^{-S_0}(-S_I)^n}{\int \mathcal{D}\phi \mathrm{e}^{-S_0}}\right]$$

($n \geq 3$) with $S_I = S_2 - \sum_q \epsilon_j \phi_q \phi_{-q}/4$ [S_2 is given by eq. (3)]. In this case, it is also shown that the predominant term around $\omega \simeq \Omega$ is proportional to $1/(\omega-\Omega)^2$, and in the calculation of higher-order terms its coefficient becomes smaller with the expansion parameter $E_C/\Omega = \sqrt{E_C/E_J}$.

Finally, we consider the case of (c). The argument for the cases of (a) and (b) is made with only the calculation of bosonic propagators. The case of (c), however, requires a calculation with fermionic degrees of freedom. It can be shown that there are a self-energy correction to $\hat{G}_k$ and a vertex correction to $Q_{1,3}(q)$ owing to the fluctuation $D_0(q)$. For example, if we calculate the self-energy correction in a one-loop approximation, the result is

$$\mathrm{Im}\hat{\Sigma}^R(\epsilon) = \frac{-\pi\rho\epsilon_c}{2\Omega}\frac{\coth\frac{\Omega}{2T} - \tanh\frac{\epsilon+\Omega}{2T}}{\sqrt{(\epsilon+\Omega)^2-\Delta^2}}\begin{pmatrix}\epsilon+\Omega & -\Delta \\ -\Delta & \epsilon+\Omega\end{pmatrix}$$

for $\epsilon > \Delta - \Omega$ or $\epsilon < -\Delta - \Omega$ and

$$\mathrm{Im}\hat{\Sigma}^R(\epsilon) = \frac{-\pi\rho\epsilon_c}{2\Omega}\frac{\coth\frac{\Omega}{2T} + \tanh\frac{\epsilon-\Omega}{2T}}{\sqrt{(\epsilon-\Omega)^2-\Delta^2}}\begin{pmatrix}\epsilon-\Omega & -\Delta \\ -\Delta & \epsilon-\Omega\end{pmatrix}$$

for $\epsilon > \Delta + \Omega$ or $\epsilon < -\Delta + \Omega$. This indicates that the self-energy is ineffective within the superconducting gap; for $\Omega/\Delta \ll 1$, the effective range of energies is small ($\Delta > \epsilon > \Delta - \Omega$ or $-\Delta < \epsilon < -\Delta + \Omega$), or for $\Omega/\Delta \approx 1$ the self-energy is proportional to $\exp(-\Omega/T)$. Therefore, these effects do not change the properties of $\hat{G}_k$ and $Q_{1,3}(q)$ used in §3, at least qualitatively.

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μs and $\Omega = 2 \times 10^{-5}$ eV. Owing to this smallness of γ_ω, although there is a large factor $1/(\omega - \Omega)^2$ in the self-energy, a self-consistent calculation [by replacing $D_0(q)$ with $D(q)$ in eq. (5), for example] makes little difference in the results except for a narrow range around $\omega = \Omega$.

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