Master's Thesis

Estimation of Value-at-Risk for bank portfolio by using Markov Chain Monte Carlo method

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Abstract

Value-at-Risk models (VaR) are powerful tools for financial risk management and are widely used by regulating authorities and market players. VaR models come from the field of "worst statistics" and help to understand the worst loss with a certain probability. Because of its relative simplicity, VaR models have become popular among risk analysists working in a banking area. After the last major financial crisis of 2007-2008, the regulators tried to modify existing models for risk assessment. VaR models were criticized by many researchers and new modified models were proposed such as conditional VaR, Expected Shortfalls (ES) and GARCH models. One of the major drawbacks of VaR models is low sensitivity to the tails of the returns' distribution. Mathematically speaking, VaR is a quantile built on the Profit & Loss distribution. Ability to accurately calculate worst outcome helps banks to keep the right amount of the allocated capital, i.e. to cover all the losses if needed and not to break the one's obligations. Moving the focus from one particular bank to the whole financial system consisting of many interconnected participants, the risks' calculation become more difficult. Regulators use the "systemic risk" term for distinguishing the risk responsible for overall system stability. Defined simply as the probability of the whole system's default after breakdown of one of the participants, the systemic risk is difficult to calculate. The main reason is complexity of the system links and causal relationships between different nodes. Different mathematical theories, including chaos and complexity theories can be applied for the understanding of

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the concept of systemic risk. However, in simple approach regular risk management tools such as VaR can be used for systemic risk estimation which is implied by Basel capital requirements, main regulation guidelines in the banking field. The problem of less efficient work with tails of the distribution, is the significant drawback of VaR models. In this work, I propose Markov Chain Monte Carlo method (MCMC) for efficient sampling from the distribution. MCMC is a powerful tool using Markov chains to create random walks while sampling from the target distribution. To prove my approach I separate two different cases: 1) one-asset VaR estimation; 2) VaR estimation from multivariate distribution for portfolios consisting of several instruments. For the first case I do sampling using Metropolis-Hastings algorithm (MH), for the second – Gibbs sampling. Result is shown for the stock from S&P 500 data set, MCMC (MH) worked almost the same with regular Monte Carlo method. For the multivariate case, I simulated a situation of 3 instruments-portfolio with known Covariance matrix under the normality assumption of the marginal distributions. After calculating the conditional probabilities, I ran Gibbs sampler and succeeded to achieve marginal distributions allowing further VaR calculation. MCMC methods proved to be applicable for VaR estimation and flexible for tackling the tail estimation problem.

Keywords:

Value-at-Risk, Markov Chain Monte Carlo, Gibbs sampler, percentile estimation

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1 Introduction

"Risk" is one of the most important concepts in finance. Financial markets fluctuate every day, and it is difficult to predict where they will go tomorrow or in one month. Even if one gets the idea of the future events, it is difficult to guess what will be the market's response to them. In the financial field there are too many players involved with their different moods and strategies. Banks are key players in these muddy waters, and they need to have guarantees of having low risks in order to continue operation and serve their clients well. At the same time, banks (hereafter by banks I mean commercial banks) are interesting to maximize their profits. Here arises the optimization problem which not so easy to solve. Good product strategy has to be combined with wise risk management. "Risk is our business", as said once Oswald Grübel, CEO at UBS. "All of life is the management of risk, not its elimination", another famous quote from Walter Wriston, former chairman of Citicorp. But what does "risk" mean, indeed? Philippe Jorion in [1] defines risk as volatility of unexpected outcomes which can be the value of assets, equity, or earning.
The Federal Reserve Board has identified six types of risk:

1. Credit risk
2. Liquidity risk
3. Market risk
4. Operational risk
5. Reputation risk
6. Legal risk

General practice assumes that banks manage these types of risk on both a transaction basis and by portfolio.
Normally banks operate with the portfolios which contain different types of financial instruments – checks (aka futures, options contracts), securities (contracts with a value), etc. The Association of Chartered Certified Accountants (ACCA) has the following definition or a financial instrument:

> *A financial instrument is any contract that gives rise to a financial asset of one entity and a financial liability or equity instrument of another entity.*

Profit of the bank in general depends on the riskiness of its portfolio [2]. Financial instruments with higher risk promise higher return with lower probability of this return taking place. But because many financial institutes perform so called accrued accounting, they achieve "reported profit" at the beginning of the possession the instrument. Only to the end of its life cycle, due to poor management and higher risks behind the instrument, bank suffers from the larger losses. Financial crisis of 2007-2008 is the good example of such situation when subprime mortgage market created a huge bubble after participating in multiple trading cycles what finally led to the huge collapse of the financial system in US and affected many economics globally.
Financial crises happen in human history regularly. Below are five of the most-devastating financial crises:

- The Credit Crisis of 1772
- The Great Depression of 1929–39
- The OPEC Oil Price Shock of 1973
- The Asian Crisis of 1997
- The Financial Crisis of 2007–08

The common thing about these turmoils is that usually it is happening after introducing or discovery a new phenomenon which revolutionary changes the society. Investors and other market players are often conservative and slowly adapt

to changes. After new things come up, they are incapable to evaluate them properly and manage risks accordingly.
The highest pick of financial crisis is investors' panic. Thomas J. Fitzpatrick IV and James B. Thomson in [3] discuss the possible consequences of the large participant failure. In Figures 1.1 and 1.2 you can see how the turmoil affected key market indicators – LIBOR, USGG3M, and TED, and how the market was affected [4].
The most dangerous consequence of financial crisis is a recession. Recession is a general slowdown in economic activity. It usually follows by a downturn in the business cycle, a reduction in the production of services and goods, and increased unemployment [5]. If recession is stated for a long time (2 or more years), it is called an economic depression. For example, economic depression happened in United States in 1930s and has got the name of Great Depression. Last depression began in 2009 in Greece. Greece's GDP dropped at almost 20%, and unemployment achieved the highest record, putting one in four Greeks out of work [6].
Financial indicators can serve as a measurement tool for predicting and preventing future economic recession [7]. Thus, we can build a model for monitoring the risks and preventing future turmoil.
The main goal of this study is to provide a framework for accurate risk estimation in the current or simulated market conditions in order to prevent financial turmoil and economic recession. Additional importance to this research is brought by recent forecasts about future recession. Recessions in modern times will happen rarely but can be more dangerous to the global economy [8].
The results of this work can find application in the bank regulation. Usually provided by central authority institutions such as Central Banks, regulation is including but not limited to injecting or removing liquidity, providing guarantees, keeping banks' accounts, storing emergency reserves, dealing with troubled assets and recapitalising the banking system. Since last financial crisis in 2007-2008, the regulation authorities are trying to improve their methods and requirement to the participants of the financial system. The most famous regulation guidelines are the Basel capital requirements. They set some limits on banks' financial indicators and benchmark investment portfolios. Current widely used set of rules is

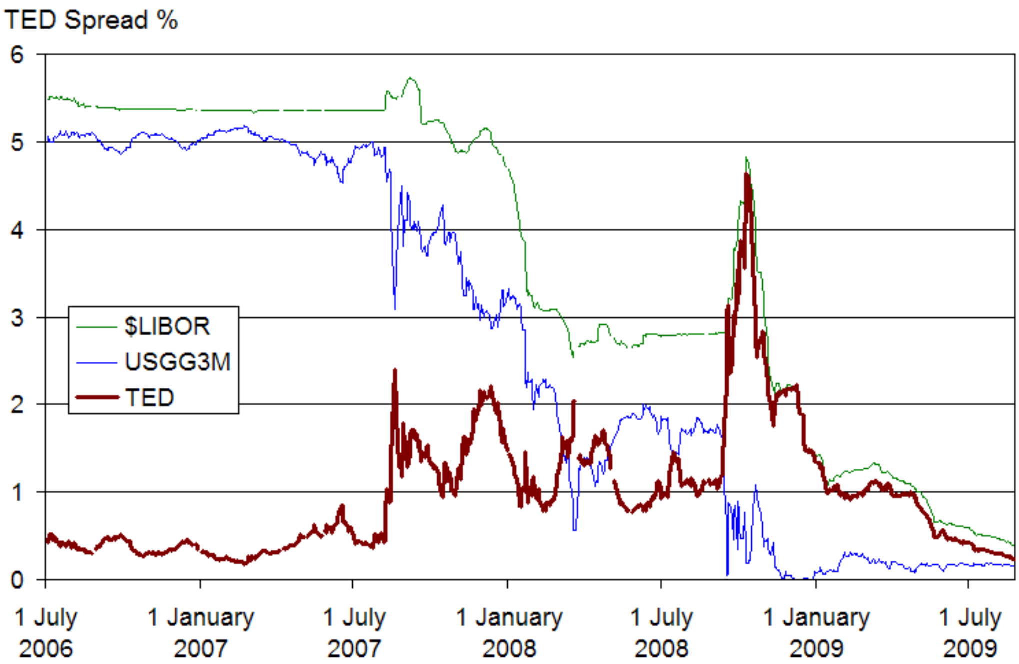


Figure 1.1: TED Spread, LIBOR, and USGG3M during financial crisis 2007-2008.

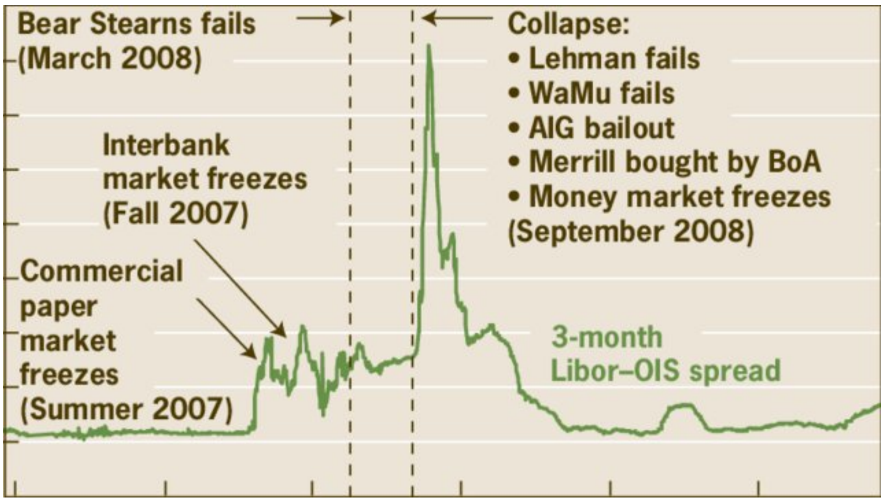


Figure 1.2: Key events and 3-month Libor-OIS spread fluctuation during the crisis 2007-2008.

called Basel III. But soon new set, Basel IV, will be introduced and gradually will be adopted by financial institutions.
In order to maintain financial stability, financial institutions and Central Banks usually perform set of simulation algorithms known as stress testing. These simulations allow to put a financial institute (for example, bank) to the strict economic condition and calculate the possibilities of its recovery or default.
Stress testing along with other techniques compose the risk management policy. One of the reasons Central banks perform them is to analyze and monitor so-called systemic risk.
The definition of systemic risk as given by Federal Reserve Governor Daniel Tarullo:

> *Financial institutions are systemically important if the failure of the firm to meet its obligations to creditors and customers would have significant adverse consequences for the financial system and the broader economy.*

In other words, systemic risk is defined as the probability that the default of one institution will make other institutions default. On practice, it is difficult to calculate systemic risk because of its pure theoretical nature. This is the reason why banks and authorities usually operate with simpler and more easy to calculate models.
One of the most famous and widely used models for risk calculation is Value-at-Risk (VaR). VaR models are a general measure of risk developed to equate risk across products and to aggregate risk on a portfolio basis. For a given portfolio, time period, and probability p, the p VaR can be defined as the maximum possible loss which can happen with probability $1 - p$ with the observed data over the given period and for given portfolio. Figure 1.3 is taken from [9] and shows the example for VaR calculation for the given returns (Profit & Loss probability density function).
In this study I mainly focus on VaR estimation using it as my main Risk Management tool. There is no standard solid approach to estimate value-at-risk for a portfolio so that it corresponds ideally to the economic background and reflects the risk of portfolio accurately. I do research into various VaR estimation models and propose new method for VaR estimation.

This thesis is organized as follows. Chapter 2 provides theoretical aspects of the risks and VaR. Chapter 3 analyzes the related work and existing literature. Chapter 4 sets the problem and main research question of this work. Chapter 5 provides methodology and research design. Chapters 6 and 7 give an overview of the results and make a conclusion.

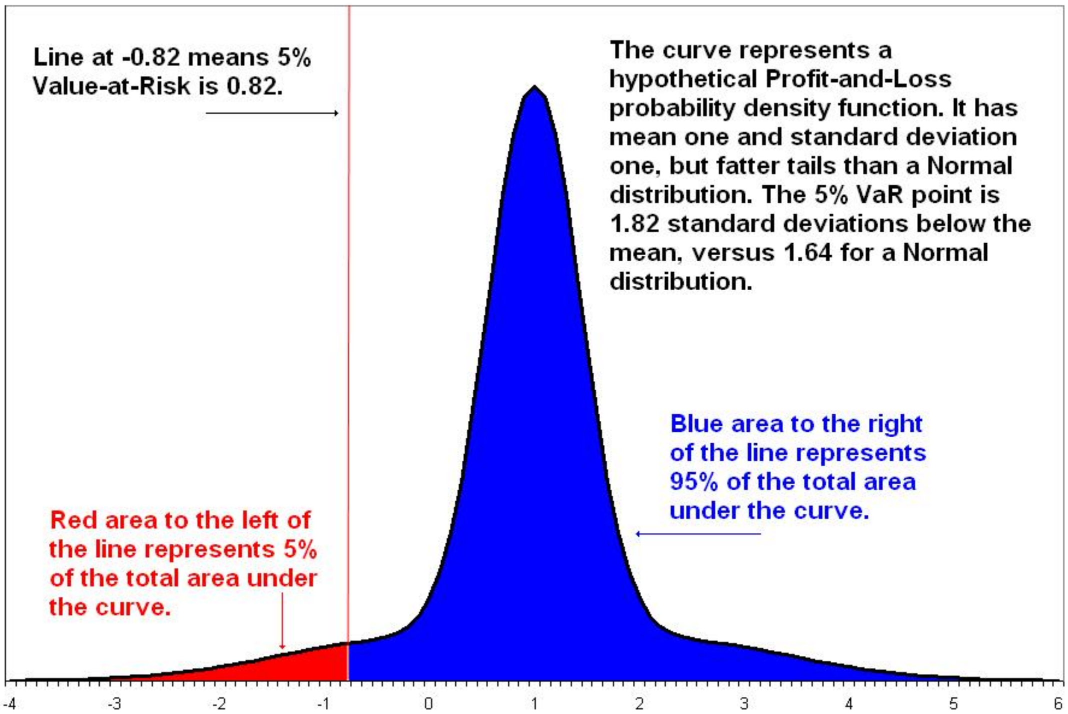


Figure 1.3: The 5% Value at Risk of a hypothetical Profit & Loss probability density function

2 Theoretical Framework

Second only to its macrostability responsibilities is the central bank's responsibility to use its authority and expertise to forestall financial crises (including systemic disturbances in the banking system) and to manage such crises once they occur.

Greenspan (1997)

In this chapter I am going to introduce some of the mathematical aspects of systemic risk and Value at Risk.
Term "systemic risk" can be applied not to the financial context only. In general, such risk is based on the network structure and complexity of the system where vulnerability of one component can lead to entire system collapse. The key distinguish of systemic risk from other types of risks is that it leads to a huge potential threat to the system while is based on complex web of interacting elements [10]. There are four major properties of systemic risk [10]:

1. Transboundary;
2. Highly interconnected leading to complex causal structures;
3. Nonlinear in the cause-effect relationships;
4. Stochastic in their effect structure.

Lucas et al. [10] separates systemic risk from the conventional risk management and classify it as dynamically changed phenomena which can be studied in such natural sciences as those of chaos, order, and self-organization, as well as methodologies of complexity science. Systemic risk can produce ripple effects beyond the domain where it occurs. The main problem is to predict whether a system will

suffer a collapse.
In this research we consider market dynamics as a set of assets values which fluctuate continuously and represent Wiener processes (geometric Brownian motions):

$$dR = -Rdt + dw \tag{2.1}$$

where R is the asset's value, and dw – Wiener process noise.
Then we can introduce interaction between fluctuations of 2 different assets' values ($R1$ and $R2$) in a single portfolio:

$$dR_1 = (-R_1 + \alpha_1 R_2)dt + dw_1 \tag{2.2}$$

$$dR_2 = (-R_2 + \alpha_2 R_1)dt + dw_2 \tag{2.3}$$

After adding more assets (financial instruments) to the portfolio we will have more complicated, multivariate Profit & Loss distribution.
In this work we consider only risk related to the portfolio. Having this assumption, we can focus on the models which have a good reputation and wide use, such as Value at Risk models. In the future research I am going to consider the transition from the static conventional model to more complex dynamic model.
Value-at-Risk models are one of the most trusted risk assessment tools for risk estimation. Bank regulation (e.g. Basel III/IV capital requirements) permit banks to use VaR to analyze potential risk within the financial institution, portfolio, or asset [11].
In general, VaR (VaR_α) is a statistic which addresses the question of when (a time period), how likely (a confidence level α) and how much (a loss amount R or loss percentage) the financial institution can loss within the financial asset. It can be formulated [1], [12] as

$$Pr(R < -VaR_\alpha) = \alpha \tag{2.4}$$

In other words, VaR is defined as the predicted worst-case loss R with a specific confidence level α (for example, 5%) over a period of time (for example, 1 day).
Standard VaR has three components:

- time period,
- confidence level (α),
- loss amount (or loss percentage) (R).

For example, if a portfolio has a one-day 1% VaR of $10,000, that means that there is a 0.01 probability that the portfolio will lose in value more than $10,000 over a one-day. In other words, a loss of $10,000 or more on this portfolio is expected on 1 day out of 100 days (because of 1% probability).
Sometimes in risk management the notation of $(1 - \alpha)$ can be used, e.g. 99% VaR instead of 1% VaR.
But VaR has one significant drawback. At some situations VaR can ignore what happens in the tails of distributions.
Consider an example where we are tossing a coin and bet on it. If head comes up, we win $100, if tail – we lose $100. Consider the bet on the fact that heads will not appear 7 times in a row. From Combinatorics we know that having them in a row can be in 1 case out of 128, e.g. ratio is 1:127. We will win more than 99.2% of time, which exceeds 99% threshold. 99% VaR is equal to zero, but we are exposed to a possible $12,700 loss [13].
To overcome the problem with tails, more advanced VaR methods were proposed such as Expected Shortfall or CoVaR [14].

3 Review of the Literature

After financial crisis of 2007-2008, it became obvious that traditional risk management tools do not work efficiently. Some researches claimed for considering new models where we can pay more attention to the correlations between the nodes across complex systems, see, for example, [16]. One of the problems is that announcing an early signal about rising in systemic risk might affect market behavior and change its dynamics. However, such indicators can be available for regulative authorities only. Together with Lucas et al. [10] they call for new models from the theory of chaos and order, and complexity science, as well as agent-based models (ABMs).
Even though systemic risk is difficult to approach, some works try to propose models for its estimation while considering correlations between bank asset portfolios and even different markets on the national level. For example, see [17].
However, I leave systemic risk modeling for a future work. In this work I am trying to approximate VaR assuming it is the main contributor to the systemic risk.
VaR in risk management is relatively new approach. It was introduced first in 1990s, but it might have been used for a long time by the companies without public announcement. After it became known to the world it has got a lot of critics because of inaccurate prediction on tails and lacking the sub-additivity property. But because of its simplicity, VaR is commonly used and said to be very popular method within the industry.
VaR can be seen as a percentile, thus we are particularly interested in the tails of the Profit & Loss distribution. One of the parametric approaches focusing on tails, is Extreme-Value theory (EVT). EVT extends the Central Limit theorem and good in measuring unexpected rare events. See, for example, [15]. But EVT is usually inaccurate for the center of distribution, that is why some researches

call this approach *semiparametric* [1].
Because VaR in some situations is weak in accurate prediction of tail events, some more advanced methods were proposed. Tobias Adrian and Markus K. Brunnermeier proposed so-called CoVaR method which stands for conditional value-at-risk [14]. Each institution contributes to the systemic risk by the difference in the CoVaR conditioned on the distressed condition and the CoVaR in the median state of the institution. This approach showed good accuracy in prediction of the financial crisis 2007-2008 for some of the stocks.
General methods for VaR estimation include methods based on historical data, such as generalized autoregressive conditional heteroskedasticity (GARCH) models [23], and modeling via copulas [24]. GARCH models have the following number of features [1]:

- Put more weight on recent information;
- Assume that variance of returns follows a predictable process;
- Provide parsimonious model with few parameters that seems to fit data well.

GARCH models have become a mainstay of time-series analysis of financial markets.
For combination of VaR forecasts, penalized quantile regressions can be used [25]. Some of the new approaches include portfolio optimization with entropic VaR [26] and even using advantage of quantum computing [27].
For this work it is particularly interesting to take a look into recent advance in using MCMC sampling for VaR estimation [28]. Though, they have shown effectiveness of MCMC comparing to other methods, choosing proposal distribution may depend on underlying data structure and may not be obvious choice. However, Guidelines for proposal distribution selection are given.
On practice, the basic methods for VaR estimation are used [29], [30]:

1. Historical method
   The historical method simply re-organizes actual historical returns, putting them in order from worst to best. It then assumes that history will repeat itself, from a risk perspective. Example of VaR calculation using historical

method is shown on Figure 3.1

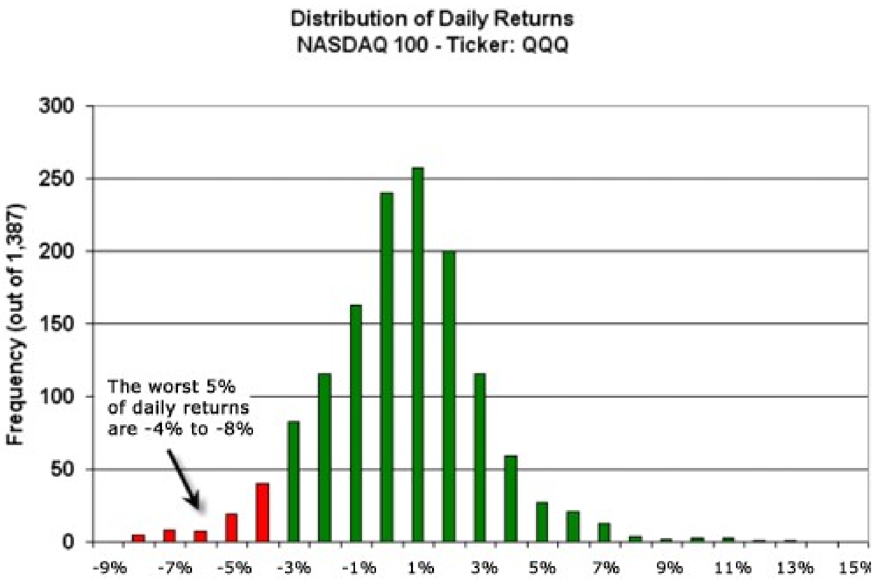


Figure 3.1: Example of Historical method for QQQ index: With 95% confidence, we expect that our worst daily loss will not exceed 4%. Source: [30]

2. Variance-Covariance (Gaussian) method
   This method assumes that stock returns are normally distributed. In other words, it requires that we estimate only two factors - an expected (or average) return and a standard deviation - which allow us to plot a normal distribution curve. Table 3.1 shows correspondence between different VaR confidence levels and low bound of Standard Deviation. On the Figure 3.2 we plot the normal curve against the same actual return data.

3. The modified Cornish-Fisher method
   This method utilizes a different distribution for the distribution of losses to better account for the possible fat-tailed nature of downside risk [31], [32]. It also provides a modified VaR calculation that takes the higher moments of non-normal distributions (skewness, kurtosis) into account through the

Confidence	# of Standard Deviations (σ)
95% (high)	- 1.65 x σ
99% (really high)	- 2.33 x σ

Table 3.1: Corresponded VaR as lower bound (#) of Standard Deviation for given Confidence

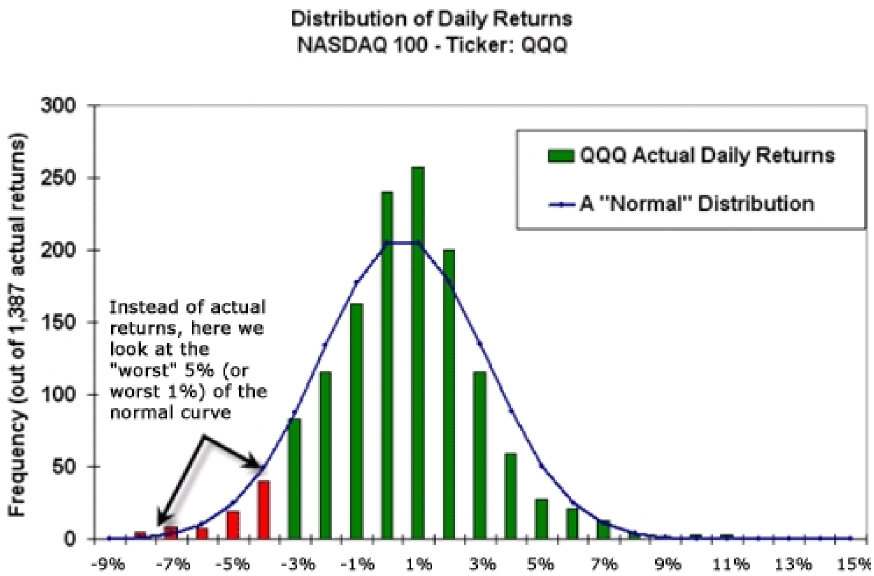


Figure 3.2: Example of Gaussian method for QQQ index. Source: [30]

use of a Cornish Fisher expansion, and collapses to standard (traditional) mean-VaR if the return stream follows a standard distribution.

4. Monte Carlo Simulation

   A Monte Carlo simulation refers to any method that randomly generates trials, but by itself does not tell us anything about the underlying methodology. Monte Carlo simulation on the QQQ based on its historical trading pattern. In our simulation example, 100 trials were conducted (see Figure 3.3). The worst five outcomes (that is, the worst 5%) were less than -15%. The Monte Carlo simulation therefore leads to the following VAR-type conclusion: with 95% confidence, we do not expect to lose more than 15% during any given month.

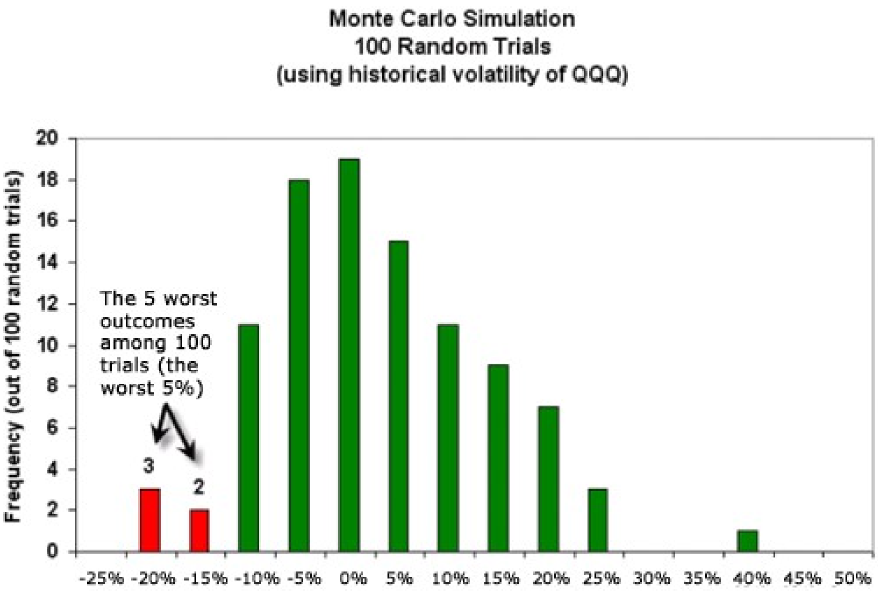


Figure 3.3: Example of Monte Carlo Simulation method for QQQ index. Source: [30]

4 Statement of Problem

In this research I am planning to find new method for VaR estimation. General problem of the VaR models is that they are poor at the tail events prediction.
Instead of building a parametric model, I will do sampling from existing historical data of the stock market prices.
Among non-parametric solutions for VaR estimation using sampling, Monte Carlo method is the most popular. However, it suffers from inefficient sampling, especially on complex, multivariate distributions.
In this work I will propose Markov Chain Monte Carlo approach and will see if it can provide more efficient sampling and give better prediction near the tails.

5 Methodology

In this study I propose Markov Chain Monte Carlo (MCMC) method for sampling from Profit & Loss distribution for consecutive VaR estimation.
Markov chain Monte Carlo (MCMC) is a method to generate samples from a (often intractable) distribution (called target distribution) by constructing a Markov chain who's stationary distribution is the desired one [18].
See example of MCMC on Figure 5.1 [20]. It shows MCMC for estimation of the posterior pdf for the Cauchy distribution parameters. The dashed curves in the top-right panel draw the results of direct computation. The solid curves are the corresponding MCMC estimates using 10,000 sample points. The left and the bottom panels show marginal distributions.
In this work the values sampled via random walks representing a Markov chain with stationary distribution equal to the target distribution of stock returns.
The original formulation of MCMC was made by Metropolis et al. (1953). Later these ideas were formulated into Metropolis-Hastings (MH) algorithm [19].
MCMC is a powerful tool for sampling from and computing expectations with respect to very complicated high-dimensional probability distributions. According to "Computing in Science & Engineering" journal, MCMC is one of the top ten most powerful mathematical algorithms of the XX century [21].
Standard Monte-Carlo method has the following form:

$$E[f(x)] = \frac{1}{n}\sum_{i=0}^{n} f(x_i) \tag{5.1}$$

$$x_i \sim p\ i.i.d.$$

But p might be too complicated to sample from which lead to inefficient or useless sampling. Possible solutions can be importance sampling or rejection sampling. But on practice it is difficult to work with high dimensional distributions (for example, to choose a proposal distribution to sample from). MCMC can be more

efficient.
Intuitively, MCMC works within the following procedure:

1. Start sampling somewhere

2. Move around looking for the region with higher probability

3. Walk around these regions

Sample decision is made according to the random walk which represents a Markov process. Generated samples consist a Markov chain, A *Markov chain* is a stochastic model describing a sequence of possible events in which the probability of each event depends only on the state attained in the previous event.
Markov chains which work in MCMC algorithm should satisfy the Ergodic theorem:

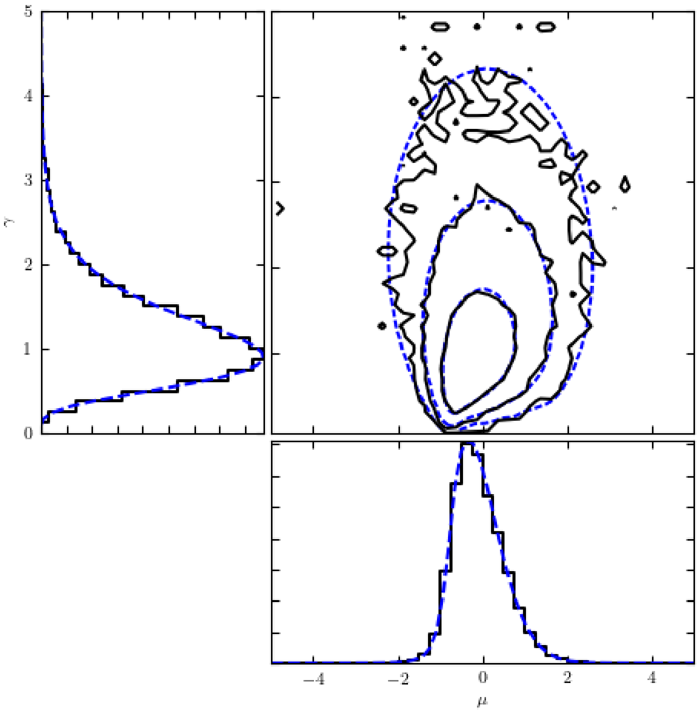


Figure 5.1: MCMC for estimation of the posterior pdf for the Cauchy distribution parameters. Source: [20]

Ergodic theorem

If $(X_0, X_1, \ldots, X_n)$ is an irreducible (time-homogeneous) discrete Markov Chain with stationary distribution π, then

$$\frac{1}{n}\sum_{i=1}^{n} f(X_i) \xrightarrow{n \to \infty} E[f(x)] \tag{5.2}$$

where $x \sim p$ for any bounded function $f : X \to R$

If, further, it is aperiodic, then

$$P(X_n = x | X_0 = x_0) \xrightarrow{n \to \infty} \pi(x), \forall x, x_0 \in X \tag{5.3}$$

I adapted MH algorithm for the task of VaR estimation.

Design of the experiment:

1. Try MCMC sampling (MH algorithm) from stock returns distribution;
2. Calculate VaR;
3. Compare with existing methods.

The above experiment is set for one-asset VaR estimation. On practice, banks operate with combined portfolios consisted of many positions. Value at risk is formulated based on so-called position vectors which indicate open positions for the portfolio with addition of some economical benchmarks.

To calculate VaRs we need to know Profit & Loss distribution. It is easier to find VaR of each marginal of the total distribution dictated by a position vector. In the banking business, analysts are often able to calculate covariances between the different components of a position vector, or, in other words, portfolio instruments. In the situations where we know conditional probabilities between the variables but cannot understand the structure of multivariate distribution, Gibbs sampling can be a useful technique.

Gibbs sampling is a modification of MCMC. Instead of sampling from the joint distribution, Gibbs sampler generates a sample from conditional distributions [33]. Consider the case of two variables (X, Y). To generate a "Gibbs sequence", first we fix the initial value for one of the variables:

$$Y_0' = y_0' \tag{5.4}$$

Then we generate new values from

$$X'_j \sim f(x|Y'_j = y'_j) \tag{5.5}$$

$$Y'_{j+1} \sim f(y|X'_j = x'_j) \tag{5.6}$$

We can obtain parameters of conditional distributions from the covariance matrix, under the normality assumption of underlying marginal distributions (and, thus, the joint distribution), using the following theorem [34].
If x is a N-dimensional vector which can be partitioned as follows

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \ with\ sizes\ \begin{bmatrix} q \times 1 \\ (N-q) \times 1 \end{bmatrix} \tag{5.7}$$

and accordingly mean μ and variance Σ are partitioned as follows

$$\mu = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} \ with\ sizes\ \begin{bmatrix} q \times 1 \\ (N-q) \times 1 \end{bmatrix} \tag{5.8}$$

$$\Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix} \ with\ sizes\ \begin{bmatrix} q \times q & q \times (N-q) \\ (N-q) \times q & (N-q) \times (N-q) \end{bmatrix} \tag{5.9}$$

then the distribution of x_1 conditional on $x_2 = a$ is multivariate normal $(x_1 \mid x_2 = a) \sim N(\hat{\mu}, \hat{\Sigma})$ where

$$\hat{\mu} = \mu_1 + \Sigma_{12}\Sigma_{22}^{-1}(a - \mu_2) \tag{5.10}$$

$$\hat{\Sigma} = \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21} \tag{5.11}$$

6 Results

For testing different VaR models we took an open data set stockdata from R package “huge” with stock prices of S&P 500 companies from 2003 to 2008 [22]. The data is collected for 1258 trading days and represents the 452 stocks’ close prices. I took stock price of eBay company for numerical calculation of VaR. The fluctuation of the price of this stock can be seen on Figure 6.1.

I calculated returns using formula

$$R = \frac{S_t - S_{t-h}}{S_{t-h}} \tag{6.1}$$

where S_t is close price at day t, and h is holding period.

Histogram of returns can be seen in Figure 6.2. Distribution of the returns depends on the chosen holding period i.e. the time for which company’s shares were on balance of the investor’s portfolio, before the day on which return is calculated. For short holding periods (1-3 days) the distribution of the returns is close to the Gaussian distribution. I decided to perform our analysis on the holding period of 40 days.

For the eBay stock we calculated VaR at 90%, 95% and 99% confidence levels using historical, Gaussian, Cornish-Fisher, Monte Carlo (MC), and MCMC methods. Ordinary Monte Carlo method depends on the size of sampling we perform. The bigger the sampling set, the better MC approximates the target distribution. I tried MC method on 100 and 500 samples generation. Figures 6.3a and 6.3b show the resulted histogram of sampling from the returns’ distribution for Monte Carlo method with sampling size of 100 and 500 respectfully.

MCMC estimation depends on the step size, number of iterations, and starting point. In this work I decided to start from “0” point. My experiments showed that if I do sampling for not very long time (for example, 100 iterations for 1298-sized data set), then MCMC often shows better result comparing to ordinary

MC. But for longer runs both methods approximate target distribution equally good. Figures 6.4a and 6.4b show the resulted histogram of sampling from the returns' distribution for the Markov Chain Monte Carlo method with sampling size of 100 and 500 respectfully.

I noticed that MCMC analyzes fat tails better and often set VaR slightly higher if there is a case. For example, look at the histograms of sampled values using both methods with 100 iterations on Figure 6.5a, and for number of iterations 500 on Figure 6.5b.

I found that when using step size = 0.12, my MCMC estimator converges well.

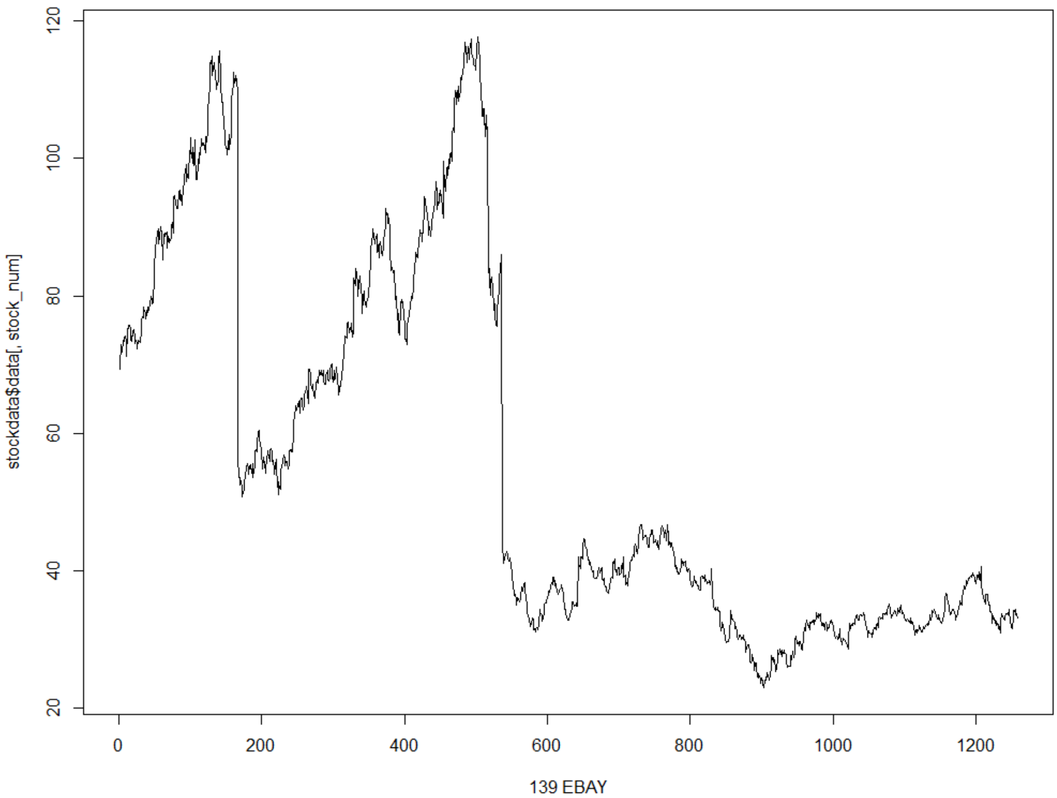


Figure 6.1: Fluctuation of the Ebay stock price.

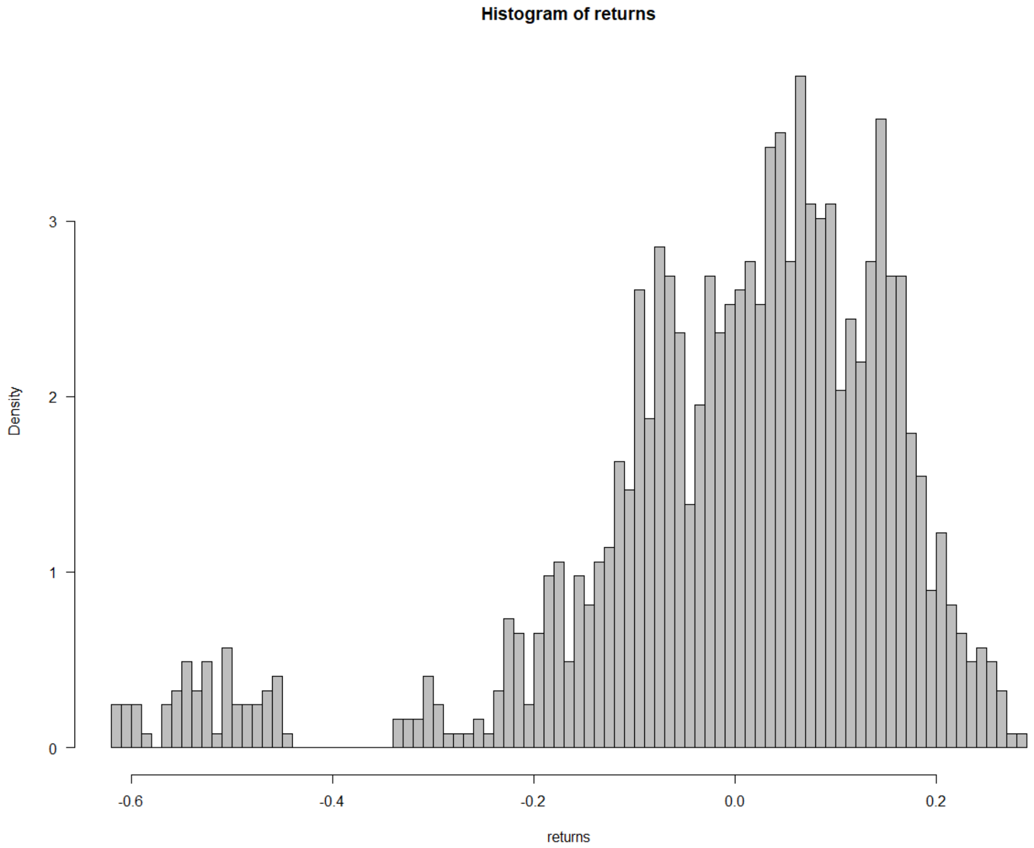


Figure 6.2: Histogram of returns for eBay stockholders with holding period of 40 days.

The MCMC convergence for 100 and 500 iterations can be seen on Figures 6.6a and 6.6b respectfully.
MCMC performance can be compared to other methods (historical, Gaussian, Cornish-Fisher, and ordinary Monte Carlo) on Figures 6.7 and 6.8 for 100 and 500 samples/iterations respectfully.

Finally, to check the Gibbs sampling for multivariate case, I assumed that bank holds a portfolio of 3 instruments. I also assumed that he Profit & Loss distribution of this portfolio sets the following Covariance matrix between the 3 variables:

$$\Sigma = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 3 & 1.5 \\ 0 & 1.5 & 4 \end{pmatrix} \tag{6.2}$$

Then we are going to sample from the joint normal distribution with zero mean and covariance matrix Σ.

$$(\Theta_1, \Theta_2, \Theta_3) \ \sim \ N(0, \Sigma) \tag{6.3}$$

To set up a Gibbs sampler with Markov chain choosing based on the conditional probabilities we use the Theorem (14), (15) to calculate

$$\Theta_1 \mid \Theta_2, \Theta_3,$$

$$\Theta_2 \mid \Theta_1, \Theta_3,$$

$$\Theta_3 \mid \Theta_1, \Theta_2.$$

It returned that

$$\mu_{1|2,3} = 0.4102564\Theta_2 - 0.1538462\Theta_3 \tag{6.4}$$

which leads us to the conclusion that

$$\Theta_1 \mid \Theta_2, \Theta_3 \ \sim \ N(0.4102564\Theta_2 - 0.1538462\Theta_3, \sqrt{1.589744}) \tag{6.5}$$

The same way I received the following expressions for other conditionals:

$$\mu_{2|1,3} = 0.5\Theta_1 + 0.375\Theta_3 \tag{6.6}$$

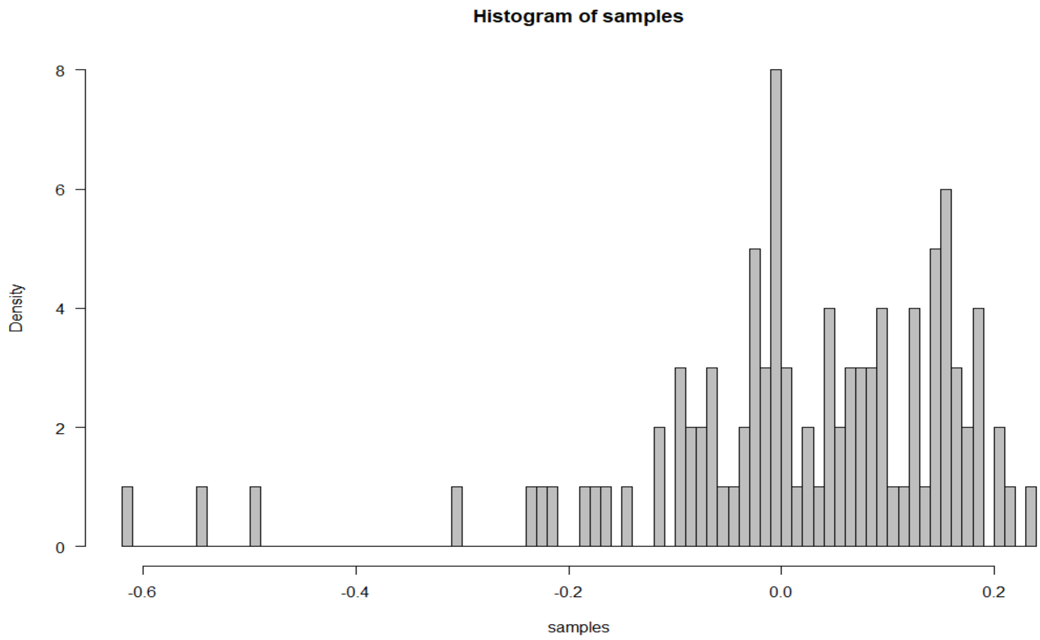
$$\Theta_2 \mid \Theta_1, \Theta_3 \ \sim \ N(0.5\Theta_1 + 0.375\Theta_3, \sqrt{1.9375}) \tag{6.7}$$

$$\mu_{3|1,2} = -0.3\Theta_1 + 0.6\Theta_2 \tag{6.8}$$

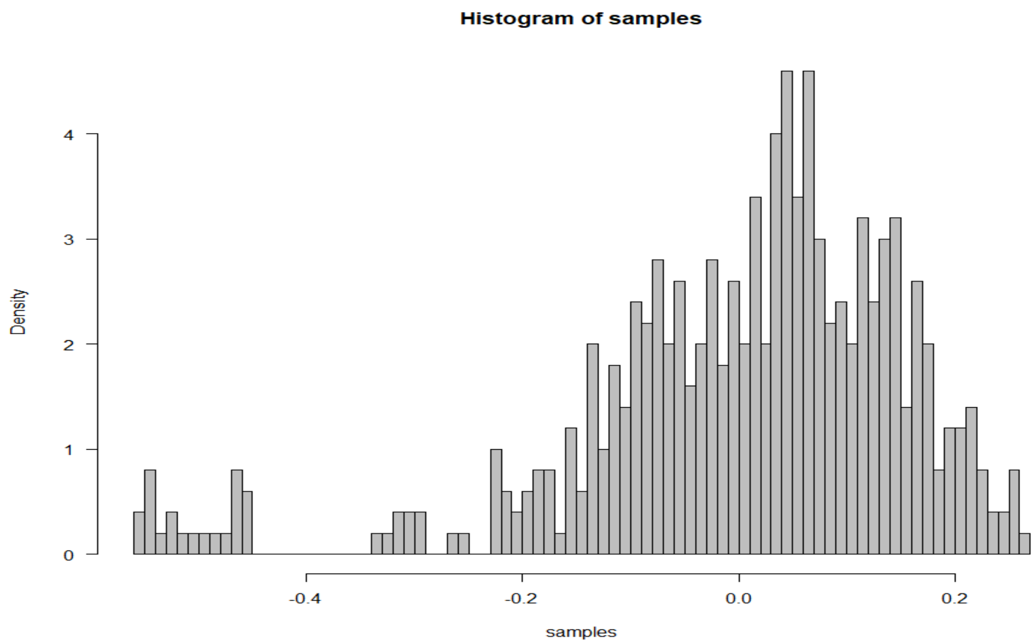
$$\Theta_3 \mid \Theta_1, \Theta_2 \ \sim \ N(-0.3\Theta_1 + 0.6\Theta_2, \sqrt{3.1}) \tag{6.9}$$

Finally, I did the sampling according to Gibbs algorithm (5.4-5.6).
From the sampling we can take a record of each marginal component and build its own distribution. You can find the according histograms on the Figures 6.9a, 6.9b, 6.9c.
After having sampled values for the marginal returns, we can easily calculate VaR for each component as a quantile.

Partly, these results were presented at the IEICE-NC2019 conference at Okinawa, in June 2019.

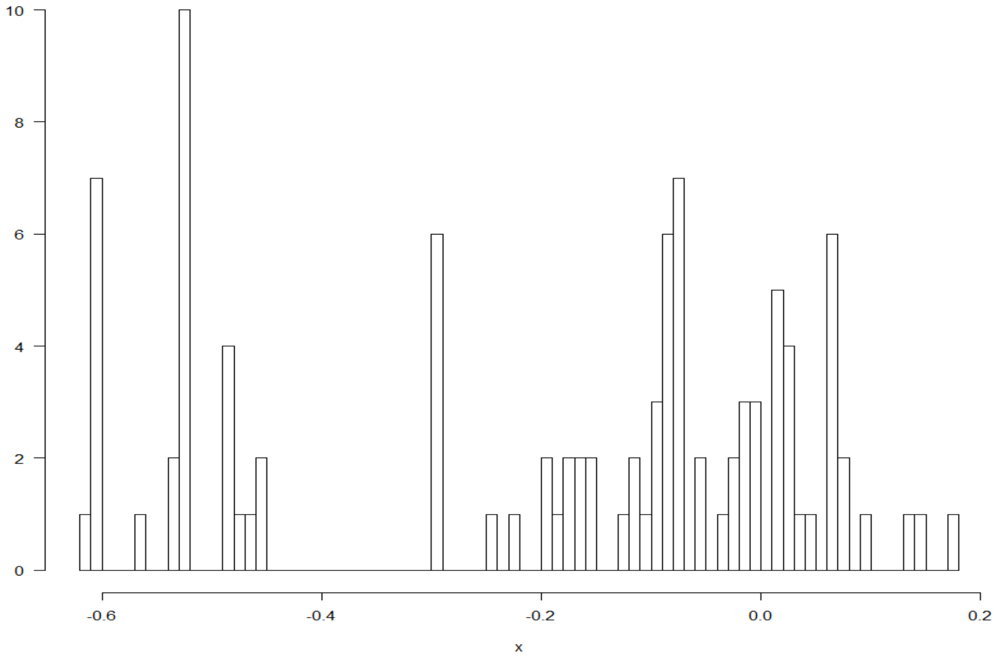


(a) Sampling size = 100

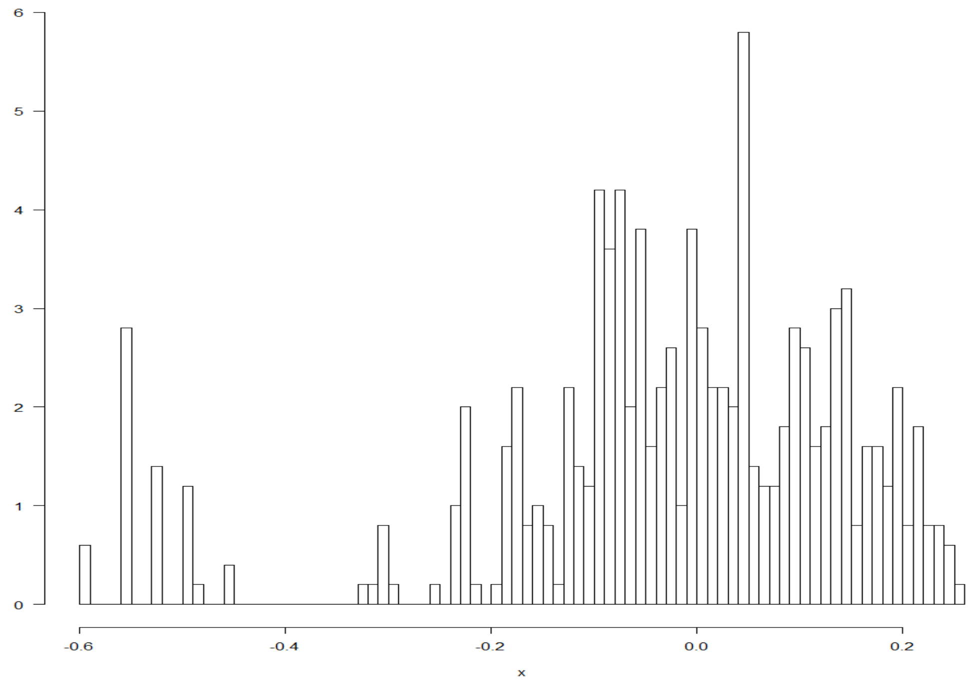


(b) Sampling size = 500

Figure 6.3: Histogram of returns after MC sampling.

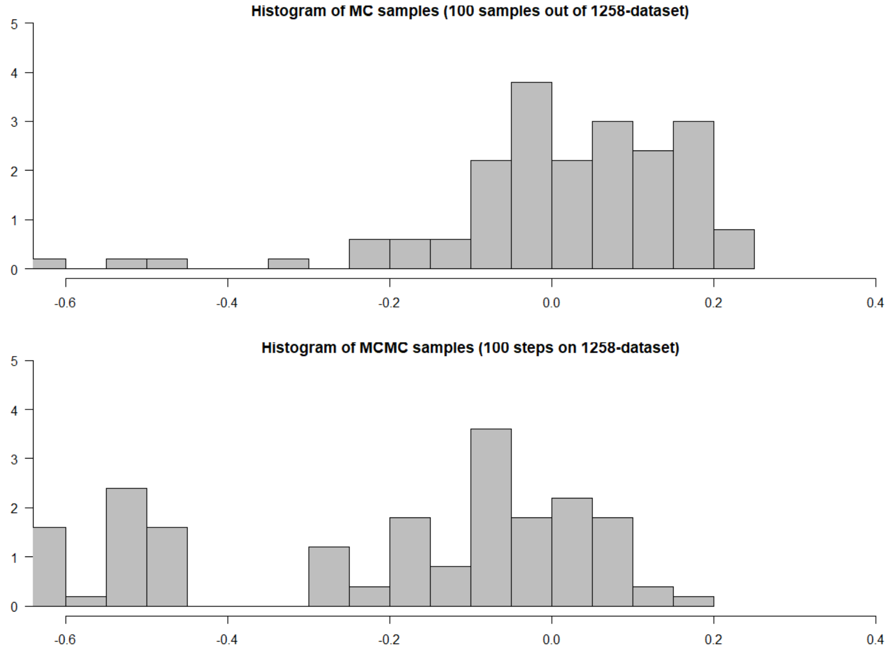


(a) Sampling size = 100

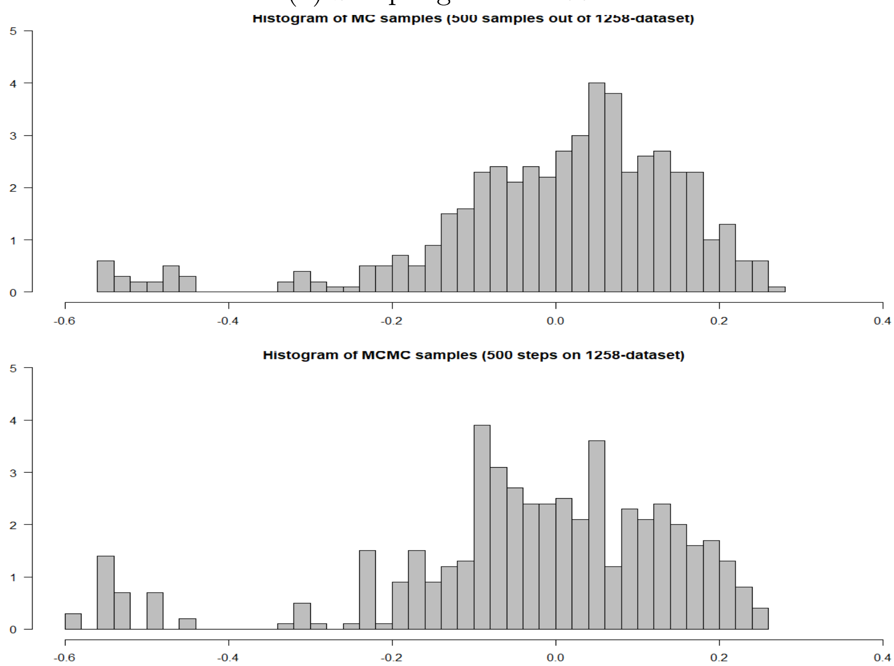


(b) Sampling size = 500

Figure 6.4: Histogram of returns after MCMC sampling.

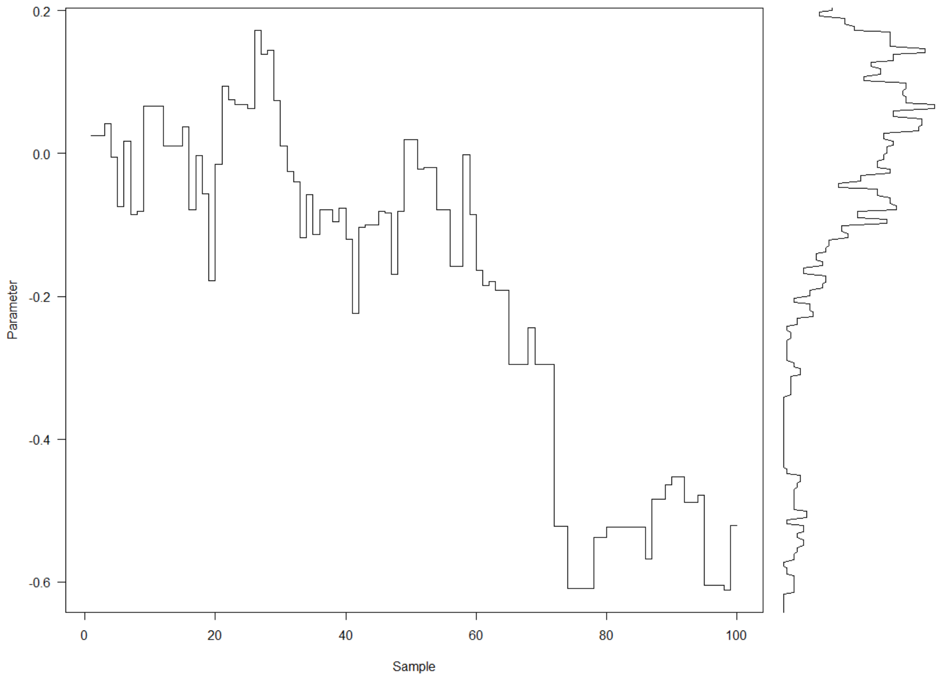


(a) Sampling size = 100

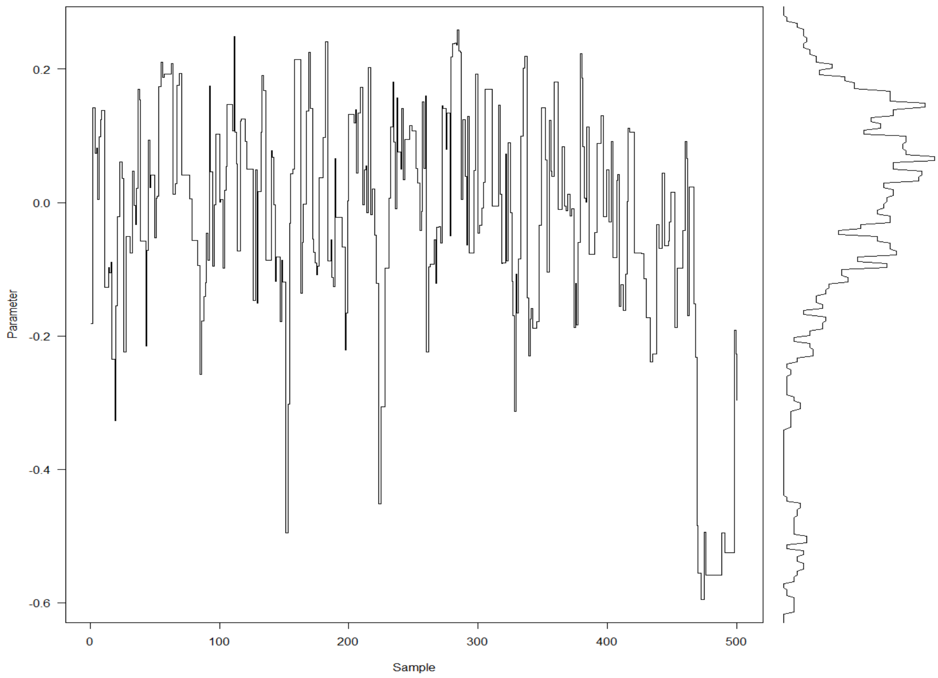


(b) Sampling size = 500

Figure 6.5: MC and MCMC methods' sample histograms.



(a) MCMC convergence for 100 iterations



(b) MCMC convergence for 500 iterations

Figure 6.6: MCMC convergence for 100 and 500 iterations.

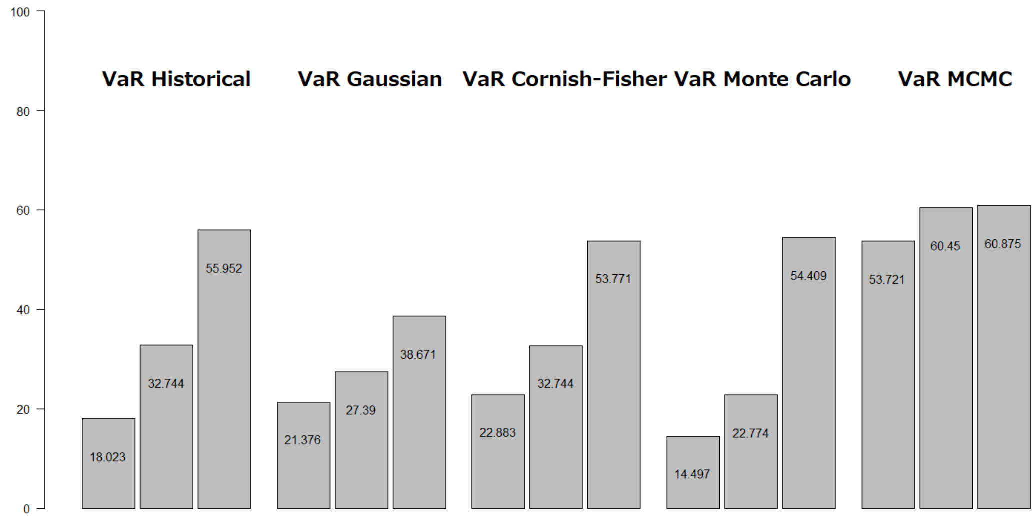


Figure 6.7: MCMC performance in comparison with other methods (historical, Gaussian, Cornish-Fisher, and ordinary Monte Carlo) for 100 samples/iterations.

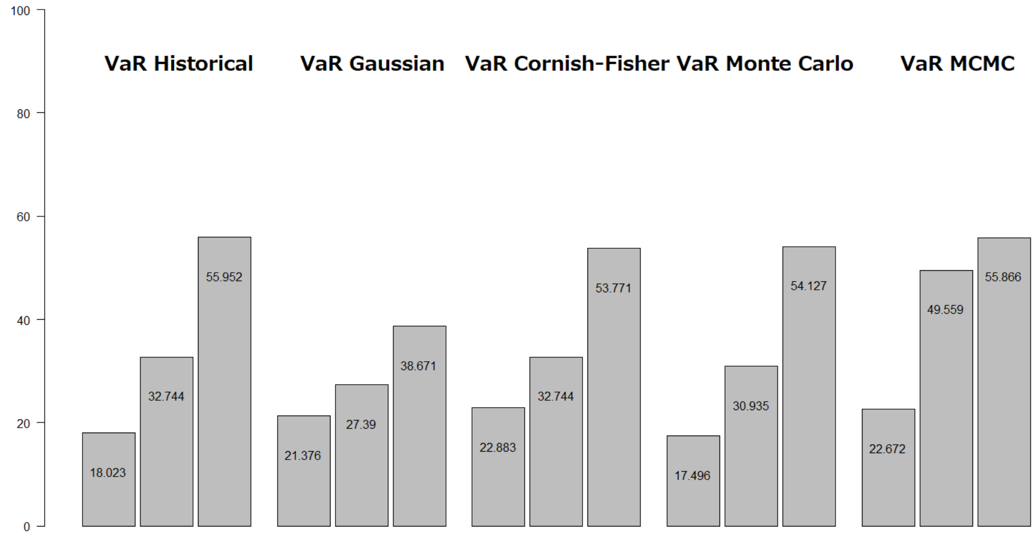
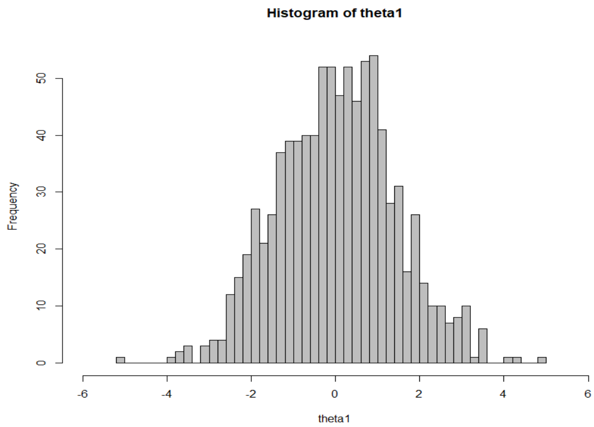
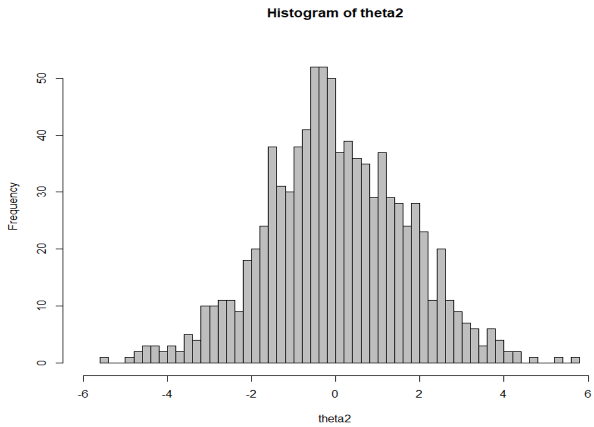


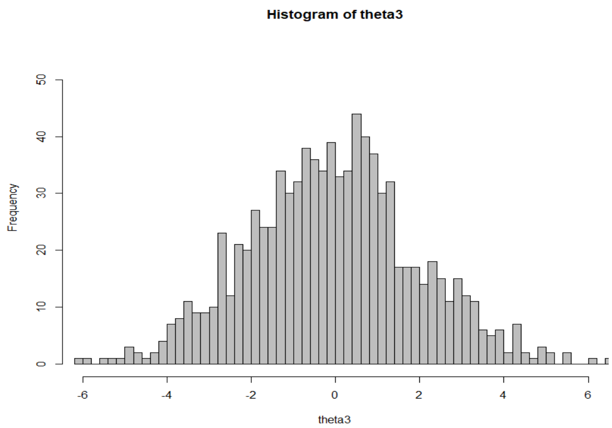
Figure 6.8: MCMC performance in comparison with other methods (historical, Gaussian, Cornish-Fisher, and ordinary Monte Carlo) for 500 samples/iterations.



(a) Histogram of Θ_1



(b) Histogram of Θ_2



(c) Histogram of Θ_3

Figure 6.9: Histogram of parameters Θ_1, Θ_2, Θ_3 which represent marginal Profit & Loss distributions for every single portfolio instrument.

7 Conclusion

Overall, we can conclude that MCMC can be used for VaR estimation. But in our experiments, the method did not show better performance compared to existing methods.
However, general methods such as historical or Gaussian can be applied to one-asset VaR calculation. In case of the structured portfolio, some modifications of MCMC such as Gibbs sampler showed the potential in VaR estimation. This application of Gibbs sampler was tested in this work. But I assumed that joint Profit & Loss distribution has normally distributed marginals. I did it for simplification of finding of conditional distributions. However, this assumption can be lifted for general MCMC application.
As part of our future work, we will consider more complicated multidimensional input based not only on one stock price fluctuation but on more complex market indicators. It is proved that MCMC sampling is effective for sampling in multidimensional space. We can apply Machine Learning method on the Profit & Loss domain. Moreover, we can consider other MCMC estimators rather than MH or Gibbs sampler.

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References

[1] Philippe, Jorion. "Value at risk: the new benchmark for managing financial risk." NY: McGraw-Hill Professional (2001).

[2] Koch, Timothy W., and S. Scott MacDonald. Bank management. Nelson Education, 2014.

[3] Fitzpatrick IV, Thomas J., and James B. Thomson. "How well does bankruptcy work when large financial firms fail? Some lessons from Lehman Brothers." Economic Commentary 23 (2011).

[4] Financial crisis 2007-2008, <https://en.wikipedia.org/wiki/Financial_crisis_of_2007%E2%80%9308>, accessed on January 6th, 2020.

[5] Spotlight, B. L. S. "The Recession of 2007-2009." US Bureau of Labor Statistics.* Staff, J., Osgood, DW, Schulenberg, JE, Bachman, JG, & Messersmith, EE (2012).

[6] UPDATE 1-Greece sinks deeper into depression in third quarter, Reuters 2012, <https://www.reuters.com/article/greece-gdp/update-1-greece-sinks-deeper-into-depression-in-third-quarter-idUSL5E8ME4WM201 accessed on January 6th, 2020.

[7] Estrella, Arturo, and Frederic S. Mishkin. "Predicting US recessions: Financial variables as leading indicators." Review of Economics and Statistics 80.1 (1998): 45-61.

[8] “Recessions have become rarer and more scary”, Robin Harding, Financial Times, September 3, 2019.

[9] Value at Risk, < https://en.wikipedia.org/wiki/Value_at_risk>, accessed on January 6th, 2020.

[10] Lucas, Klaus, Ortwin Renn, and Carlo Jaeger. "Systemic Risks: Theory and Mathematical Modeling." Advanced Theory and Simulations 1.11 (2018): 1800051.

[11] King, Peter, and Heath Tarbert. "Basel III: an overview." Banking & Financial Services Policy Report 30.5 (2011): 1-18.

[12] Acharya, Viral V., et al. "Measuring systemic risk." The Review of Financial Studies 30.1 (2017): 2-47

[13] Einhorn, David, and Aaron Brown. "Private profits and socialized risk." Global Association of Risk Professionals 42 (2008): 10-26.

[14] Adrian, Tobias, and Markus K. Brunnermeier, 2011, CoVaR, Staff Reports 348, Federal Reserve Bank of New York.

[15] Baran, Jaroslav, and Jiri Witzany. "A comparison of EVT and standard VaR estimations." Available at SSRN 1768011 (2011).

[16] Battiston, Stefano, et al. "Complexity theory and financial regulation." Science 351.6275 (2016): 818-819.

[17] Lehar, Alfred. "Measuring systemic risk: A risk management approach." Journal of Banking & Finance 29.10 (2005): 2577-2603.

[18] Dagpunar, John S. Simulation and Monte Carlo: With applications in finance and MCMC. John Wiley & Sons, 2007.

[19] Hastings, W. Keith. "Monte Carlo sampling methods using Markov chains and their applications." (1970): 97-109.

[20] “MCMC for the Cauchy distribution”, < https://www.astroml.org/book_figures/chapter5/fig_cauchy_mcmc.html>, accessed on January 6th, 2020.

[21] Dongarra, Jack, and Francis Sullivan. "Guest editors' introduction: The top 10 algorithms." Computing in Science & Engineering 2.1 (2000): 22.

[22] Zhao, Tuo, et al. "The huge package for high-dimensional undirected graph estimation in R." Journal of Machine Learning Research 13.Apr (2012): 1059-1062.

[23] Engle, Robert. "GARCH 101: The use of ARCH/GARCH models in applied econometrics." Journal of economic perspectives 15.4 (2001): 157-168.

[24] Fantazzini, Dean. "Dynamic copula modelling for value at risk." Frontiers in Finance and Economics 5.2 (2008): 72-108.

[25] Bayer, Sebastian. "Combining Value-at-Risk forecasts using penalized quantile regressions." Econometrics and statistics 8 (2018): 56-77.

[26] Ahmadi-Javid, Amir, and Malihe Fallah-Tafti. "Portfolio optimization with entropic value-at-risk." European Journal of Operational Research (2019).

[27] Woerner, Stefan, and Daniel J. Egger. "Quantum risk analysis." npj Quantum Information 5.1 (2019): 15.

[28] Koike, Takaaki, and Mihoko Minami. "Estimation of risk contributions with MCMC." Quantitative Finance (2019): 1-19.

[29] Bohdalová, Mária. "A comparison of value-at-risk methods for measurement of the financial risk." The Proceedings of the E-Leader (2007): 1-6.

[30] An Introduction to Value at Risk (VAR), < `https://www.investopedia.com/articles/04/092904.asp`>, accessed on January 6th, 2020.

[31] Zangari, Peter. A VaR Methodology for Portfolios that include Options. 1996. RiskMetrics Monitor, First Quarter, 4-12.

[32] Laurent Favre and Jose-Antonio Galeano. Mean-Modified Value-at-Risk Optimization with Hedge Funds. Journal of Alternative Investment, Fall 2002, v 5.

[33] Casella, George, and Edward I. George. "Explaining the Gibbs sampler." The American Statistician 46.3 (1992): 167-174.

[34] Eaton, Morris L. "Multivariate statistics: a vector space approach." JOHN WILEY & SONS, INC., 605 THIRD AVE., NEW YORK, NY 10158, USA, 1983, 512 (1983).

Publication List

[1] Igor Zavialov, Kazushi Ikeda, MCMC for Value-at-Risk estimation, IEICE Tech. Rep., vol. 119, no. 88, NC2019-11, pp. 41-44, June 2019