One sufficient termination condition for some iterative soft-decision algorithms decoding binary linear block codes

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Abstract

Concerning the optimality of candidate codewords generated in the procedure of soft-decision decoding binary linear Block codes, here we give some further discussions on the sufficient conditions proposed by Kasami et.al. A more effective sufficient condition is also obtained based on using the distance profile of the codes. This sufficient condition can serve as an effective temination condition for many iterative soft-decision algorithms decoding binary linear block codes.

Key words and phrases: Binary linear block code, Iterative soft-decision decoding, Maximum likelihood decoding.

1 Introduction

To reduce the decoding complexity of soft-decision decoding binary linear block codes, many iterative soft-decision decoding algorithms were proposed. In almost of these algorithms, a simple decoder is employed to generate a series of candidate codewords, and the optimal codeword will be among these generated canditate codewords certainly or with very high probability. However, it is difficult to know the suitable number of the candidate codewords which need to be generated. In general, after the generating of a new candidate codewode a test on optimality of it is implemented to determine whether to terminate the decoding iteration process or continue. Many test conditions on the optimality of generated canditate codewords were derived, such as those proposed by Taipale and Pursley [1], Kaneko et.al. [2] and Moorthy et.al. [3]. Recently, Kasami et.al in [4] have investigated some sufficient conditions on the optimality of a canditate codeword based on some generated candidate codewords and partial or complete knowldge of the weight profile of the code. The applications of these sufficient conditions have shown they are much more effective than their opponents. Here we shall do some further studies on these sufficient conditions, a new sufficient condition is also obtained.

Suppose a binary block code C of length N with distance profile $W_C = \{0, w_1, w_2, \ldots\}$, $0 < w_1 < w_2 < \cdots$, is used for error control over some memoryless channel. Let $\boldsymbol{z}= (\boldsymbol{z}_1, \ldots, \boldsymbol{z}_N)$ be the sequence obtained by performing the hard-decision on a received sequence $\boldsymbol{y}= (\boldsymbol{y}_1, \ldots, \boldsymbol{y}_N)$, and suppose $\boldsymbol{r}= (\boldsymbol{r}_1, \ldots, \boldsymbol{r}_N)$ is the relative reliability sequence, where $\boldsymbol{r}_i \geq 0$, $i = 1, \ldots, N$. For convenient, we assume that the reliability sequence are ordered according to the following increasing order

$$\boldsymbol{r}_i \leq \boldsymbol{r}_j, \quad \text{for } 1 \leq i < j \leq N. \tag{1}$$

For positive integer k, let B^k denote the set of all binary sequence with length k. Let $\boldsymbol{u}_i$ denote the i-th component of a sequence $\boldsymbol{u}$. Let $d_H(\boldsymbol{u},\boldsymbol{v})$ denote the hamming distance between two sequences $\boldsymbol{u}$ and $\boldsymbol{v}$ in B^N. For a sequence $\boldsymbol{u}\in B^N$, let $\overline{D}_1(\boldsymbol{u})$ denote the set of indices i with $\boldsymbol{u}_i \neq \boldsymbol{z}_i$, let $n(\boldsymbol{u})$ denote the cardinality of $\overline{D}_1(\boldsymbol{u})$, i.e. $n(\boldsymbol{u}) = |\overline{D}_1(\boldsymbol{u})| = d_H(\boldsymbol{u},\boldsymbol{z})$. The following

$$L(\boldsymbol{u}) = \sum_{i\in \overline{D}_1(\boldsymbol{u})} \boldsymbol{r}_i \tag{2}$$

is called the correlation discrepancy of $\boldsymbol{u}$. The purpose of the maximum likelihood decoding(MLD) is to find a code $\boldsymbol{c}_{\text{opt}}$ in the code C such that

$$L(\boldsymbol{c}_{\text{opt}}) = \min_{\boldsymbol{c}\in C} L(\boldsymbol{c}). \tag{3}$$

Such a codeword $\boldsymbol{c}_{\text{opt}}$ is called the optimal codeword(or the optimal MLD codeword).

In a specified set of candidate codewords R, a codeword $\boldsymbol{c}$ is said to be the best if $L(\boldsymbol{c})$ is the minimum among R. Clearly,we have the following lemma.

Lemma 1 *Let R be a set of candidate codewords. If $\boldsymbol{u}_{best}$ is the best codeword in R, then $\boldsymbol{u}_{best}$ is the optimal codeword $\boldsymbol{c}_{opt}$ if and only if*

$$L(\boldsymbol{u}_{best}) \leq \min_{\boldsymbol{u}\in C\setminus R} L(\boldsymbol{u}), \tag{4}$$

In the iterative decoding algorithms proposed in [5-7], a low-weight subtrellis search around a codeword $\boldsymbol{u}$ is performed to find the best codeword in $\overline{C}_{w_{j_0}}(\boldsymbol{u}) = \{\boldsymbol{v} \in C : d_H(\boldsymbol{v}, \boldsymbol{u}) \leq w_{j_0}\}$ for small $j_0(j_0 = 1$ or $2)$.

Let $\boldsymbol{u}^1, \ldots, \boldsymbol{u}^h$ be some reference codewords. Here we shall consider to give a new sufficient condition for the best codeword $\boldsymbol{u}_{\text{best}}$, which is obtained by implementing the low-weight subtrellis search upon the set of codewords $\cup_{i=1}^{h}\overline{C}_{w_{j_0}}(\boldsymbol{u}^i)$, is the optimal codeword $\boldsymbol{c}_{\text{opt}}$.

For a sequence $\alpha \in B^h$, let $\overline{D}_\alpha$ and n_α be defined as

$$\overline{D}_\alpha = \bigcap_{i=1}^{h} \overline{D}_{\alpha_i}(\boldsymbol{u}^i), \tag{5}$$

$$n_\alpha = |\overline{D}_\alpha|, \tag{6}$$

where $\overline{D}_0(\boldsymbol{u}^i) = \{1, \ldots, N\}\backslash\overline{D}_1(\boldsymbol{u}^i)$. If α and β are different sequeces in B^h, clearly, we have

$$\overline{D}_\alpha \cap \overline{D}_\beta = \emptyset(\text{the empty set}). \tag{7}$$

For $\boldsymbol{u} \in B^N$, let

$$q_\alpha(\boldsymbol{u}) = |\overline{D}_1(\boldsymbol{u}) \cap \overline{D}_\alpha|. \tag{8}$$

Clearly, for every $\alpha \in B^h$, we have

$$0 \le q_\alpha(\boldsymbol{u}) \le n_\alpha, \tag{9}$$

and

$$|\overline{D}_0(\boldsymbol{u}) \cap \overline{D}_\alpha| = n_\alpha - q_\alpha(\boldsymbol{u}). \tag{10}$$

Furthermore, with respect to (7), it is not difficult to show that

$$d_H(\boldsymbol{u}, \boldsymbol{u}^i) = |\overline{D}_0(\boldsymbol{u}) \cap \overline{D}_1(\boldsymbol{u}^i)| + |\overline{D}_1(\boldsymbol{u}) \cap \overline{D}_0(\boldsymbol{u}^i)|, \tag{11}$$

$$|\overline{D}_0(\boldsymbol{u}) \cap \overline{D}_1(\boldsymbol{u}^i)| = |\overline{D}_0(\boldsymbol{u}) \cap (\cup_{\alpha\in B^h, \alpha_i=1}\overline{D}_\alpha)| = \sum_{\alpha\in B^h, \alpha_i=1} (n_\alpha - q_\alpha(\boldsymbol{u})), \tag{12}$$

$$|\overline{D}_1(\boldsymbol{u}) \cap \overline{D}_0(\boldsymbol{u}^i)| = |\overline{D}_1(\boldsymbol{u}) \cap (\cup_{\alpha\in B^h, \alpha_i=0}\overline{D}_\alpha)| = \sum_{\alpha\in B^h, \alpha_i=0} q_\alpha(\boldsymbol{u}), \tag{13}$$

$$\sum_{\alpha\in B^h, \alpha_i=1} n_\alpha = n(\boldsymbol{u}^i). \tag{14}$$

So we have

$$\sum_{\alpha\in B^h} (-1)^{\alpha_i} q_\alpha(\boldsymbol{u}) = d_H(\boldsymbol{u}, \boldsymbol{u}^i) - n(\boldsymbol{u}^i), \quad i = 1, \ldots, h, \tag{15}$$

Let q and q' be k-tuple over integers, we write $q \le q'$ if $q_i \le q'_i$, for $i = 1, \ldots, k$. Let X be a subset of $\{1, 2, \ldots, N\}$. For an integer j, let $X^{(j)}$ denote the set of j smallest integers in X if $1 \le j \le |X|$, the empty set $\emptyset$ if $i \le 0$, and X if $j > |X|$, respectively.

Let Q^h denote the set of all the 2^h-tuple $\boldsymbol{q}$'s over nonnegative integers. For a subset Q' of Q^h, we say $\boldsymbol{q} \in Q'$ is minimal in Q' if there is no $\boldsymbol{q}' \in Q'$ such that $\boldsymbol{q}' \leq \boldsymbol{q}$, and $\boldsymbol{q}' \neq \boldsymbol{q}$. Let $Q'_{\min}$ denote the set of minimal tuples in Q'.

For a h-tuple τ over nonnegative integers, let $Q(\tau)$ denote the set of those $\boldsymbol{q} \in Q^h$ which satisfy

$$\sum_{\alpha \in B^h} (-1)^{\alpha_i} q_\alpha = \tau_i + w_{j_0+1} - n(\boldsymbol{u}^i), \quad i = 1, \ldots, h. \tag{16}$$

Write

$$\overline{Q}(\tau) = \{q \in Q(\tau) : q_\alpha \leq n_\alpha\}, \tag{17}$$

$$Q[\tau] = \bigcup_{\tau' \geq \tau} Q(\tau'), \tag{18}$$

$$\overline{Q}[\tau] = \bigcup_{\tau' \geq \tau} \overline{Q}(\tau'). \tag{19}$$

Let $T(\tau)$ denote the set of those h-tuple τ' over nonnegative integers which satisfy

$$0 \leq \tau' \leq \tau, \ \tau' \neq \tau, \tag{20}$$

$$\tau'_i + w_{j_0+1} \in W_C, \ i = 1, \ldots, h. \tag{21}$$

Write

$$\overline{Q}_\tau = \bigcup_{\tau' \in T(\tau)} \overline{Q}(\lambda'). \tag{22}$$

Let

$$L_1^\tau = \min_{\boldsymbol{q} \in \overline{Q}[\tau]_{\min}} \sum_{i \in \cup_{\alpha \in B^h} \overline{D}_\alpha^{(q_\alpha)}} \boldsymbol{r}_i, \tag{23}$$

$$L_2^\tau = \min_{\boldsymbol{q} \in \overline{Q}_\tau} \sum_{i \in \cup_{\alpha \in B^h} \overline{D}_\alpha^{(q_\alpha)}} \boldsymbol{r}_i. \tag{24}$$

Then we have the following conclusion.

Lemma 2 *Let $\boldsymbol{u}_{best}$ be the best codeword obtained by implementing the low-weight subtrellis search upon the set of codewords $\cup_{i=1}^h \overline{C}_{w_{j_0}}(\boldsymbol{u}^i)$. If there is a h-tuple τ over nonnegative integers such that*

$$L(\boldsymbol{u}_{best}) \leq \min\{L_1^\tau, L_2^\tau\}, \tag{25}$$

then $\boldsymbol{u}_{best}$ is the optimal codeword $\boldsymbol{u}_{opt}$.

Proof: Suppose that $\boldsymbol{u}\in C\setminus \cup_{i=1}^{h} \overline{C}_{w_{j_0}}(\boldsymbol{u}^i)$. Clearly, we have

$$d_H(\boldsymbol{u}, \boldsymbol{u}^i) \in \{w_{j_0+1}, w_{j_0+2}, \ldots\},\ i = 1, \ldots, h. \tag{26}$$

Let $\boldsymbol{q}$ denote the 2^h-tuple with $\boldsymbol{q}_\alpha = \boldsymbol{q}(\boldsymbol{u})_\alpha$, $\alpha \in B^h$. Then from (15) we know that either $\boldsymbol{q}\in \overline{Q}[\tau]$ or $\boldsymbol{q}\in \overline{Q}_\tau$. Thus according to (1), Lemma 1 and the definition of $\overline{Q}[\tau]_{\min}$, we know that (25) is a sufficient condition for the $\boldsymbol{u}_{\text{best}}$ is the optimal codeword $\boldsymbol{u}_{\text{opt}}$. △△

If $h = 1$, then the right side of (25) is simply the same for every $\tau \geq 0$. It is clear that the right side of (25) is increasing when τ inceases. If we choose $\tau = 0$, then (25) is the sufficient conditions obtained by T.Kasami et.al. in [4]. So, if $h \geq 2$, then Lemma 2 gives a more effective sufficient condition. This condition can apply to many iterative soft-decision algorithms for decoding binary linear codes as an effective termination condition. The computation of L_1^τ of the cases $h \leq 3$ has been considered in [?]. Below we shall give a method of computing L_1^τ in the case $h = 4$. Concerning the computing of L_2^τ, the cases $h = 2$ and $h = 3$ are also considered.

In section 2, we give some general disscusions on the set $Q[\tau]_{\min}$. Some general conditions for a 2^h-tuple $\boldsymbol{q}$ over nonnegative integers to belong to $Q[\tau]_{\min}$ are obtained. In section 3, all the elements of $Q[\tau]_{\min}$ in the case $h = 4$ are determined. In section 4, a algorithm for computing L_1^τ in the case $h = 4$, based on the disscusions in section 2 and section 3, is proposed. In section 5, we consider the computation of L_2^τ in the cases of $h = 2$ and $h = 3$ by investigating the set of $\overline{Q}_\tau$.

2 The general discussion on the set $Q[\tau]_{\min}$

For a sequence $\alpha \in B^h$, let $(-1)^\alpha$ denote the h-tuple for which the i-th component is $(-1)^{\alpha_i}, i = 1, \ldots, h$, let $\overline{\alpha}$ denote the sequence in B^h with $\overline{\alpha}_i = 1 - \alpha_i$, $i = 1, \ldots, h$. Let Γ_0 denote the set of all h-tuples γ over nonnegative integers for which $\gamma_i = 0$ holds for at least one i with $1 \leq i \leq h$.

For $\boldsymbol{q}\in Q[\tau]$, let

$$\Delta(\boldsymbol{q}) = \sum_{\alpha\in B^h} \boldsymbol{q}_\alpha(-1)^\alpha - \delta, \tag{27}$$

where $\delta = (\delta_1, \ldots, \delta_h)$ with

$$\delta_i = \tau_i + w_{j_0+1} - n(\boldsymbol{u}^i),\ i = 1, \ldots, h. \tag{28}$$

Let $SP_{\boldsymbol{q}}$ denote the support set of $\boldsymbol{q}$, i.e.

$$SP_{\boldsymbol{q}} = \{\alpha \in B^h : \boldsymbol{q}_\alpha \geq 1\}. \tag{29}$$

Then, from the definition of $Q[\tau]_{\min}$, we can get the following Lemma 3 easily

Lemma 3 *If $\boldsymbol{q} \in Q[\tau]_{\min}$, then $(1, \ldots, 1) \notin SP_{\boldsymbol{q}}$ and $\Delta(q) \in \Gamma_0$. If $\boldsymbol{q} \in Q[\tau]$, and only one of the component of $\Delta(\boldsymbol{q})$, say the i-th one, is equal to 0, then $\boldsymbol{q} \in Q_{\min}$ if and only if $SP_{\boldsymbol{q}} \subset \{\alpha \in B^h : \alpha_i = 0\}$.*

Lemma 4 *If $\boldsymbol{q} \in Q[\tau]$, then $\boldsymbol{q} \in Q[\tau]_{\min}$ if and only if $\sum_{\alpha \in B^h} \boldsymbol{q}'_\alpha (-1)^\alpha \not\leq \Delta(\boldsymbol{q})$ holds for every $\boldsymbol{q}' \neq 0$, $0 \leq \boldsymbol{q}' \leq \boldsymbol{q}$.*

Proof: It is clear that

$$\begin{aligned}
\boldsymbol{q} \in Q_{\min} &\iff \delta \not\leq \sum_{\alpha \in B^h} \boldsymbol{q}'_\alpha (-1)^\alpha \text{ for any } \boldsymbol{q}' \neq \boldsymbol{q}, 0 \leq \boldsymbol{q}' \leq \boldsymbol{q} \\
&\iff \sum_{\alpha \in B^h} (\boldsymbol{q}_\alpha - \boldsymbol{q}'_\alpha)(-1)^\alpha \not\leq \sum_{\alpha \in B^h} \boldsymbol{q}_\alpha (-1)^\alpha - \delta \text{ for any } \boldsymbol{q}' \neq \boldsymbol{q}, 0 \leq \boldsymbol{q}' \leq \boldsymbol{q} \\
&\iff \sum_{\alpha \in B^h} \boldsymbol{q}'_\alpha (-1)^\alpha \not\leq \Delta(\boldsymbol{q}) \text{ for any } \boldsymbol{q}' \neq 0, 0 \leq \boldsymbol{q}' \leq \boldsymbol{q}.
\end{aligned}$$

△△

If $\Xi \subset B^h$ and $\gamma \in \Gamma_0$ satisfy

$$\sum_{\alpha \in \Xi} \xi_\alpha (-1)^\alpha \not\leq \gamma \tag{30}$$

for any nonnegative integers ξ_α with $\sum_{\alpha \in \Xi} \xi_\alpha \geq 1$, then we call Ξ a $m(\gamma)$-set.

Let Ξ is a $m(\gamma)$-set. If either $\alpha \in \Xi$ or $\bar{\alpha} \in \Xi$ holds for any $\alpha \in B^h$, or in other words, $|\Xi| = 2^{h-1}$, then we call Ξ a $M(\gamma)$-set.

Clearly, if Ξ is a $m(\gamma)$-set, then Ξ is a $m(0)$-set and $(1, \ldots, 1) \notin \Xi$.

Lemma 5 *If $h \leq 5$, $\Xi \subset B^h$, $\gamma \in \Gamma_0$, then Ξ is a $m(\gamma)$-set if and only if $(-1)^\alpha \not\leq \gamma$ and $(-1)^\alpha + (-1)^{\alpha'} \not\leq \gamma$ for any $\alpha, \alpha' \in \Xi$.*

Proof: If $h = 1$ or $h = 2$, the lemma is obvious. Below we assume that $3 \leq h \leq 5$.

At first, we consider the case of $\gamma_i \leq 1$, $i = 1, 2, \ldots, h$. From Lemma 3, we can only show the case of $\sum_{j=1}^h \gamma_j \leq h - 2$. Assume that $(-1)^\alpha \not\leq \gamma$ and $(-1)^\alpha + (-1)^{\alpha'} \not\leq \gamma$ for any $\alpha, \alpha' \in \Xi$, but Ξ is not a $m(\gamma)$-set. Then there are some $\alpha^{(i)}$'s in Ξ such that $\sum_{i=1}^k (-1)^{\alpha^{(i)}} \leq \gamma$, $k \geq 3$. Let $\lambda = \sum_{i=1}^k (-1)^{\alpha^{(i)}}$. If there is some i such that $\lambda_i = 1$, then the other components of λ must be odd integers and there are at least two j's such that $\lambda_j \leq -1$. Hence $\sum_{i=1}^{h} \lambda_i \leq 1$ because of $h \leq 5$, and so the number of -1 in $\{(-1)^{\alpha_j^{(i)}} : 1 \leq i \leq k, 1 \leq j \leq h\}$ is not smaller than $kh/2 - 1$. Thus there exists at least an i' such that the number of -1 in $\{(-1)^{\alpha_j^{(i')}} : 1 \leq j \leq h\}$ is not smaller than $h/2 - 1/k$. Let σ denote the number of 1 in $\{(-1)^{\alpha_j^{(i')}} : 1 \leq j \leq h\}$. With respect to $h \leq 5, k \geq 3$ and $[h/2 + 1/k] \leq 2$ we know $\sigma \leq 2$. Clearly, we also have $\sigma > 0$.

If $\sigma = 2$, assume that $(-1)^{\alpha_{j_1}^{(i')}} = (-1)^{\alpha_{j_2}^{(i')}} = 1$ and $(-1)^{\alpha_j^{(i')}} = -1$, for $j \neq j_1, j_2$. From $(-1)^{\alpha^{(i')}} \not\leq \gamma$ we know that $\gamma_{j_1} + \gamma_{j_2} \leq 1$. Moreover, for any $i \neq i'$, from $(-1)^{\alpha^{(i)}} + (-1)^{\alpha^{(i')}} \not\leq \gamma$, we have $(-1)^{\alpha_{j_1}^{(i)}} + (-1)^{\alpha_{j_2}^{(i)}} \geq 0$. So, we get

$$2 = (-1)^{\alpha_{j_1}^{(i')}} + (-1)^{\alpha_{j_2}^{(i')}} \leq \sum_{i=1}^{k} \left((-1)^{\alpha_{j_1}^{(i)}} + (-1)^{\alpha_{j_2}^{(i)}} \right) = \gamma_{j_1} + \gamma_{j_2} \leq 1.$$

This is a contradiction.

If $\sigma = 1$, we can also deduce a contradiction easily.

Now we consider the case $\gamma_i \geq 2$ holds for some i's. Without loss of the generality, we suppose that $\gamma_j \leq 1$, $1 \leq j \leq k$, and $\gamma_j \geq 2$, $k < j \leq h$. Writting $\gamma_r = (\gamma_1, \ldots, \gamma_k)$ and

$$\Xi_r = \{\alpha_r : \alpha_r = (\alpha_1, \ldots, \alpha_k), \alpha \in \Xi\}.$$

If $(-1)^\alpha \not\leq \gamma$ and $(-1)^\alpha + (-1)^{\alpha'} \not\leq \gamma$ for any $\alpha, \alpha' \in \Xi$, then $(-1)^{\alpha_r} \not\leq \gamma_r$ and $(-1)^{\alpha_r} + (-1)^{\alpha'_r} \not\leq \gamma_r$ for any $\alpha_r, \alpha'_r \in \Xi_r$. By the above proof, we know that Ξ_r is a $m(\gamma_r)$-set, and so Ξ is a $m(\gamma)$-set.

$\triangle\triangle$

Lemma 6 *Let $\alpha^{(i)} \in B^h$, $i = 1, 2, 3, 4$, satisfy $(-1)^{\alpha^{(i)}} + (-1)^{\alpha^{(j)}} \not\leq 0$ for any $1 \leq i, j \leq 4$. Then there are nonnegative integers $\xi_1, \xi_2, \xi_3, \xi_4$, such that*

$$\xi_1(-1)^{\alpha^{(1)}} + \xi_2(-1)^{\alpha^{(2)}} + \xi_3(-1)^{\alpha^{(3)}} + \xi_4(-1)^{\alpha^{(4)}} \leq 0, \tag{31}$$

if and only if $h \geq 6$, $\xi_1 = \xi_2 = \xi_3 = \xi_4$,

$$(-1)^{\alpha_i^{(1)}} + (-1)^{\alpha_i^{(2)}} + (-1)^{\alpha_i^{(3)}} + (-1)^{\alpha_i^{(4)}} \leq 0 \tag{32}$$

for any $1 \leq i \leq h$, and there are $i_1, \ldots, i_6$, such that

$$\{(\alpha_{i_j}^{(1)}, \alpha_{i_j}^{(2)}, \alpha_{i_j}^{(3)}, \alpha_{i_j}^{(4)}) : j = 1, \ldots, 6\} = \{\beta \in B^4 : \beta_1 + \beta_2 + \beta_3 + \beta_4 = 2\}. \tag{33}$$

Proof: Assume that there are nonnegative integers $\xi_1, \xi_2, \xi_3, \xi_4$, such that (31) holds. From $(-1)^{\alpha^{(1)}} + (-1)^{\alpha^{(2)}} \not\leq 0$, we know there is at least one index i_{12} such that $\alpha^{(1)}_{i_{12}} = \alpha^{(2)}_{i_{12}} = 0$. So from

$$\xi_1(-1)^{\alpha^{(1)}_{i_{12}}} + \xi_2(-1)^{\alpha^{(2)}_{i_{12}}} + \xi_3(-1)^{\alpha^{(3)}_{i_{12}}} + \xi_4(-1)^{\alpha^{(4)}_{i_{12}}} \leq 0,$$

we know $\xi_1 + \xi_2 \leq \xi_3 + \xi_4$. Similarly, we can get $\xi_1 + \xi_3 \leq \xi_2 + \xi_4$, $\xi_1 + \xi_4 \leq \xi_2 + \xi_3$, $\xi_2 + \xi_3 \leq \xi_1 + \xi_4$, $\xi_2 + \xi_4 \leq \xi_1 + \xi_3$, $\xi_3 + \xi_4 \leq \xi_1 + \xi_2$. Hence $\xi_1 = \xi_2 = \xi_3 = \xi_4$ and (32) hold for any $1 \leq i \leq h$. Furthermore, we know that $\alpha^{(3)}_{i_{12}} = \alpha^{(4)}_{i_{12}} = 1$, and there are $i_1, \ldots, i_6$ such that (33) holds, and $h \geq 6$.

On the other hand, assume that (32) and (33) hold for some $i_1, \ldots, i_6$. It is clear that $(-1)^{\alpha^{(i)}} + (-1)^{\alpha^{(j)}} \not\leq 0$ for any $1 \leq i, j \leq 4$, and

$$(-1)^{\alpha^{(1)}} + (-1)^{\alpha^{(2)}} + (-1)^{\alpha^{(3)}} + (-1)^{\alpha^{(4)}} \leq 0.$$

i.e. (31) holds for $\xi_1 = \xi_2 = \xi_3 = \xi_4 = 1$. △△

According to this lemma, we can see that Lemma 5 can not be generalized to the cases $h \geq 6$ directly.

From Lemma 3, Lemma 4 and Lemma 5, we can get the following corollary easily.

Corollary 1 *If $h \leq 5$, $\boldsymbol{q} \in Q[\tau]$, then $\boldsymbol{q} \in Q[\tau]_{\min}$ if and only if $\Delta(\boldsymbol{q}) \in \Gamma_0$ and $SP_{\boldsymbol{q}}$ is a $m(\Delta(\boldsymbol{q}))$-set.*

Lemma 7 *1. If Ξ be a $m(0)$-set, then Ξ must be a subset of some $M(0)$-set.*
2. If $h \leq 5$, $\gamma \in \Gamma_0$, Ξ is a $m(\gamma)$-set, then Ξ must be a subset of some $M(\gamma)$-set.

Proof: 1. Assume in the contrary that there is a $m(0)$-set Ξ such that

$$\xi'(-1)^{\beta} + \sum_{\alpha \in \Xi} \xi'_\alpha (-1)^\alpha \leq 0$$

and

$$\xi''(-1)^{\overline{\beta}} + \sum_{\alpha \in \Xi} \xi''_\alpha (-1)^\alpha \leq 0$$

hold for some $\beta \in B^h$ and nonnegative integers $\xi', \xi'', \xi'_\alpha, \xi''_\alpha$, $\alpha \in \Xi$, $\xi' + \xi'' \geq 1$. Then we have

$$\sum_{\alpha \in \Xi} (\xi' \xi''_\alpha + \xi'' \xi'_\alpha)(-1)^\alpha \leq 0,$$

this contradicts that Ξ is a $m(0)$-set.

2. Let $h \leq 5$, $\gamma \in \Gamma_0$, Ξ be a $m(\gamma)$-set.

At first, we consider the case of $\gamma_i \leq 1$, $i = 1, 2, \ldots, h$. From $-(-1)^\gamma \leq \gamma$, we can see easily that $\Xi \cup \{\gamma\}$ is a $m(0)$-set. So by the above proof, there is a $M(0)$-set Ξ' such that $\Xi \cup \{\gamma\} \subset \Xi'$. Furthermore, for any $\alpha, \alpha' \in \Xi'$, from $(-1)^\alpha + (-1)^{\alpha'} \not\leq 0$ we know that there is at least one index i such that $\alpha_i = \alpha'_i = 0$, and so $(-1)^\alpha \not\leq \gamma$ and $(-1)^\alpha + (-1)^{\alpha'} \not\leq \gamma$ for any $\alpha, \alpha' \in \Xi'$. By Lemma 5 we know that Ξ' must be a $M(\gamma)$-set which contains Ξ.

Now we consider the case that $\gamma_i \geq 2$ holds for some i's. Without loss of the generality, we assume that $\gamma_j \leq 1$, $1 \leq j \leq k$, and $\gamma_j \geq 2$, $k < j \leq h$. Writting $\gamma_r = (\gamma_1, \ldots, \gamma_k)$ and

$$\Xi_r = \{\alpha_r : \alpha_r = (\alpha_1, \ldots, \alpha_k), \alpha \in \Xi\}.$$

Then from Lemma 5, we know Ξ_r must be a $m(\gamma_r)$-set. Let Ξ'_r be a $M(\gamma_r)$-set which contains Ξ_r. Then the set

$$\{\alpha \in B^h : (\alpha_1, \ldots, \alpha_k) \in \Xi'_r\}$$

must be a $M(\gamma)$-set which contains Ξ. △△

Lemma 8 *If $h \leq 5$, Ξ is a $M(0)$-set, $\gamma \in \Gamma_0$, $\zeta = \{i : \gamma_i \leq 1, 1 \leq i \leq h\}$, then Ξ is a $M(\gamma)$-set if and only if there is a $\alpha' \in \Xi$ such that $\gamma_i = \alpha'_i$ for every $i \in \zeta$ and*

$$\Xi = \bigcup_{\alpha'' \in \Xi} \{\alpha \in B^h : \alpha_i = \alpha''_i, i \in \zeta\}. \tag{34}$$

Proof: Without loss of the generality, we suppose that $\zeta = \{1, \ldots, k\}$.

Assume that Ξ is a $M(\gamma)$-set. From Lemma 5, we know that

$$\Xi_r = \{\alpha_r : \alpha_r = (\alpha_1, \ldots, \alpha_k), \alpha \in \Xi\}$$

must be a $m(\gamma_r)$-set, where $\gamma_r = (\gamma_1, \ldots, \gamma_k)$.

Furthermore, from $|\Xi| = 2^{h-1}$ and

$$\Xi \subset \Xi' = \bigcup_{\alpha' \in \Xi_r} \{\alpha \in B^h : \alpha_i = \alpha'_i, i = 1, \ldots, k\},$$

we know that $|\Xi_r| \geq 2^{k-1}$. So that Ξ_r must be a $M(\gamma_r)$-set and $\Xi = \Xi'$. Moreover, according to $(-1)^{\overline{\gamma}_r} \leq \gamma_r$, we know $\overline{\gamma}_r \notin \Xi_r$, and then $\gamma_r \in \Xi_r$.

On the other hand, assume there is a $\alpha' \in \Xi$ such that $\gamma_i = \alpha'_i$, $i = 1, \ldots, k$, and

$$\Xi = \bigcup_{\alpha'' \in \Xi} \{\alpha \in B^h : \alpha_i = \alpha''_i, i = 1, \ldots, k\}.$$

We have that $\alpha \in \Xi$ holds provided $\alpha \in B^h$ and there is some $\alpha'' \in \Xi_r$ such that $\alpha_i = \alpha''_i$, $i = 1, \ldots, k$. Thus, with respect to Ξ is $m(0)$-set, we know that Ξ_r is a $m(0)$-set. Moreover, from $\gamma_r \in \Xi_r$, we know Ξ_r is a $m(\gamma_r)$-set. Hence Ξ is $m(\gamma)$-set. Then, with respect to that $|\Xi| = 2^{h-1}$, we know Ξ is a $M(\gamma)$-set. $\triangle\triangle$

For $h = 4$, we write

$$C_i = \{\alpha \in B^4 : \alpha_i = 0\}, \quad D_i = \{\alpha \in B^4 : \sum_{j \neq i} \alpha_j \leq 1\}, 1 \leq i \leq 4,$$

$$E_i = \{\alpha \in B^4 : \sum_{j=1}^{4} \alpha_j \leq 1\} \cup \{\alpha \in B^4 : \sum_{j=1}^{4} \alpha_j = 2, \alpha_i = 0\}, 1 \leq i \leq 4.$$

For $h = 5$, we write $\Theta_1 = \{\alpha \in B^5 : \sum_{j=1}^{5} \alpha_j \leq 1\}$, $\Theta_2 = \{\alpha \in B^5 : \sum_{j=1}^{5} \alpha_j = 2\}$, and

$$\begin{gathered} F = \Theta_1 \cup \Theta_2, \quad G_i = \{\alpha \in B^5 : \alpha_i = 0\}, 1 \leq i \leq 5, \\ H_i = \Theta_1 \cup \{\alpha \in \Theta_2 : \alpha_i = 0\} \cup \{\alpha \in B^5 : \bar{\alpha} \in \Theta_2, \alpha_i = 0\}, 1 \leq i \leq 5, \\ I_{ijk} = \{\alpha \in B^5 : \alpha_i + \alpha_j + \alpha_k \leq 1\}, 1 \leq i < j < k \leq 5, \\ J^l_{ijk} = \{\alpha \in B^5 : \alpha_i + \alpha_j + \alpha_k + \alpha_l \leq 1\} \cup \{\alpha \in B^5 : \alpha_i + \alpha_j + \alpha_k = 2, \alpha_l = 0\}, \\ 1 \leq i < j < k \leq 5, 1 \leq l \leq 5, l \neq i, j, k, \\ K_{\beta\beta'} = \Theta_1 \cup \{\alpha \in \Theta_2 : \alpha \neq \beta, \beta'\} \cup \{\bar{\beta}, \bar{\beta}'\}, \beta, \beta' \in \Theta_2, \beta_i = \beta'_i = 1, \text{ for some } i. \end{gathered}$$

Theorem 1 *1. If $h = 4$, then all the $M(0)$-sets are*

$$C_i, D_i, E_i, \ i = 1, 2, 3, 4.$$

2. If $h = 5$, then all the $M(0)$-sets are

$$\begin{gathered} F, G_i, H_i (1 \leq i \leq 5), \\ I_{ijk}, J^l_{ijk} (1 \leq i < j < k \leq 5, 1 \leq l \leq 5, l \neq i, j, k), \\ K_{\beta\beta'} (\beta, \beta' \in \Theta_2, \beta_i = \beta'_i = 1, \text{ for some } i). \end{gathered}$$

Proof: 1. Assume $h = 4$. One hand, if $\Xi \in \{C_i, D_i, E_i : i = 1, 2, 3, 4\}$, and $\alpha, \alpha' \in \Xi$, it is clear that there exist at least one i with $1 \leq i \leq 4$ such that $\alpha_i = \alpha'_i = 0$. So, $(-1)^\alpha \not\leq 0$ and $(-1)^\alpha + (-1)^{\alpha'} \not\leq 0$ for any $\alpha, \alpha' \in \Xi$, and from Lemma 5, we know Ξ is a $M(0)$-set.

On the other hand, we suppose that Ξ is a $M(0)$-set. If there is an i_0 such that $\alpha_{i_0} = 0$ for all $\alpha \in \Xi$ or $\alpha'_{i_0} = 0$, $\alpha'_i = 1$, $i \neq i_0$ for some $\alpha' \in \Xi$, then $\Xi = C_{i_0}$. Assume that there are $\alpha^{(i)} \in \Xi$ such that $\alpha^{(i)}(i) = 1$ for $i = 1, 2, 3, 4$, and $\sum_{j=1}^{4} \alpha_j \leq 2$ for every

$\alpha \in \Xi$. Let Φ denote the set of the elements $\alpha \in \Xi$ with $\sum_{j=1}^{4} \alpha_j = 2$. From $|\Xi| = 2^3$, we know $|\Phi| \geq 3$. Without loss of the generality, we assume that $(1,1,0,0),(1,0,1,0) \in \Phi$. It is clear that in Φ any other one must belong to $\{(0,1,1,0),(1,0,0,1)\}$. So, from $(-1)^{(0,1,1,0)} + (-1)^{(1,0,0,1)} = 0$, we know that $\Xi \in \{E_4, D_1\}$.

2. Assume $h = 5$. One hand, similar to the case of $h = 4$, if Ξ is one of the sets

$$F, G_i, H_i (1 \leq i \leq 5),$$
$$I_{ijk}, J_{ijk}^{l} (1 \leq i < j < k \leq 5, 1 \leq l \leq 5, l \neq i,j,k),$$
$$K_{\beta\beta'} (\beta, \beta' \in \Theta_2, \beta_i = \beta'_i = 1, \text{ for some } i),$$

then we can show easily that Ξ is a $M(0)$-set.

On the other hand, we suppose that Ξ is a $M(0)$-sets. If there are a $\alpha \in \Xi$ satisfies $\sum_{i=1}^{5} \alpha_i = 4$, say concretely, $\alpha_i = 0$, $\alpha_j = 1$, $j \neq i$, then $\Xi = G_i$ must hold. Now we suppose that $\sum_{i=1}^{5} \alpha_i \leq 3$ for every $\alpha \in \Xi$. Let $\Re$ denote the set of the sequences $\alpha \in \Xi$ with $\sum_{j=1}^{5} \alpha_j = 3$.

If $\Re$ is a empty set, then $\Xi = F$.

If $\Re = \{\alpha\}$, then $\beta = \bar{\alpha} \notin \Xi$, and $\Xi = K_{\beta\beta}$.

If $\Re = \{\alpha, \alpha'\}$, then $\beta = \bar{\alpha}, \beta' = \bar{\alpha}' \notin \Xi$, and $\beta_i = \beta'_i = 1$ for some i. So we have $\Xi = K_{\beta\beta'}$.

If $|\Re| = 3$, without loss of the generality, we assume that $(1,1,1,0,0),(1,1,0,1,0) \in \Re$. Then the other one must belong to

$$\{(1,1,0,0,1),(0,1,1,1,0),(1,0,1,1,0)\}.$$

This is equivalent to say that

$$\Xi \in \{I_{345}, J_{134}^{5}, J_{123}^{5}\}.$$

If $|\Re| \geq 4$, arguing as in the case of $|\Re| = 3$, with respect to

$$(-1)^{(1,1,0,0,1)} + (-1)^{(0,1,1,1,0)} \leq 0, \ (-1)^{(1,1,0,0,1)} + (-1)^{(1,0,1,1,0)} \leq 0,$$

we know $\Xi \in \{H_i : 1 \leq i \leq 5\}$. △△

3 The determination of $Q[\tau]_{\min}$ in the case $h = 4$

In this section, we shall give all the 2^h-tuples $\boldsymbol{q}$ in $Q[\tau]_{\min}$ for the case $h = 4$.

Lemma 9 *Let λ be a 4-tuple over integers, its components have same odd-even property. Then the equation*

$$\sum_{\alpha \in B^4} \boldsymbol{q}_\alpha (-1)^\alpha = \lambda \quad (35)$$

have some nonnegative integral solutions $\boldsymbol{q}$ *such that*
(i). $SP_{\boldsymbol{q}} \subset C_i$ *if and only if*

$$\lambda_i \geq \max_{j \neq i}\{|\lambda_j|\}, \tag{36}$$

(ii). $SP_{\boldsymbol{q}} \subset D_i$ *if and only if*

$$\sum_{j \neq i} \lambda_j \geq \max_{1 \leq j \leq 4}\{|\lambda_j|\}, \tag{37}$$

(iii). $SP_{\boldsymbol{q}} \subset E_i$ *if and only if*

$$\sum_{j=1}^{4} \lambda_j \geq 0, \ \ \lambda_i + \min_{j \neq i}\{\lambda_j\} \geq 0, \tag{38}$$

Proof: Without loss of the generality, we suppose that $i = 4$.

(i). One hand, if the equation (35) have some solutions $\boldsymbol{q}$ such that $SP_{\boldsymbol{q}} \subset C_4$, then from

$$(-1)^{\alpha_4} \geq \max_{1 \leq j \leq 3}\{|(-1)^{\alpha_j}|\},$$

for any $\alpha \in C_4$, we know that $\lambda_4 \geq \max_{1 \leq j \leq 3}\{|\lambda_j|\}$.

On the other hand, we assume $\lambda_4 \geq \max_{1 \leq j \leq 3}\{|\lambda_j|\}$. Without loss of the generality, we suppose $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq 0$. Then $\boldsymbol{q}_{0000} = (\lambda_4 + \lambda_3)/2$, $\boldsymbol{q}_{0010} = (\lambda_1 - \lambda_2)/2$, $\boldsymbol{q}_{0110} = (\lambda_2 - \lambda_3)/2$, $\boldsymbol{q}_{1110} = (\lambda_4 - \lambda_1)/2$, $\boldsymbol{q}_{1100} = \boldsymbol{q}_{1000} = \boldsymbol{q}_{0100} = \boldsymbol{q}_{1010} = 0$, is a solution.

(ii). One hand, if the equation (35) have some solutions $\boldsymbol{q}$ such that $SP_{\boldsymbol{q}} \subset D_4$, then from

$$\sum_{1 \leq j \leq 3} (-1)^{\alpha_j} \geq \max_{1 \leq j \leq 4}\{|(-1)^{\alpha_j}|\},$$

for any $\alpha \in D_4$, we know that $\sum_{1 \leq j \leq 3} \lambda_j \geq \max_{1 \leq j \leq 4}\{|\lambda_j|\}$.

On the other hand, we assume $\sum_{1 \leq j \leq 3} \lambda_j \geq \max_{1 \leq j \leq 4}\{|\lambda_j|\}$. It is sufficient if we give a $\boldsymbol{q}$ such that $SP_q \subset D_4$, $\sum_{\alpha \in D_4} q_\alpha = \lambda_1 + \lambda_2 + \lambda_3$, and $\lambda'_i = \lambda_i$ for $i = 1, 2, 3$, where

$$(\lambda'_1, \lambda'_2, \lambda'_3, \lambda'_4) = \sum_{\alpha \in D_4} q_\alpha (-1)^\alpha.$$

Clearly, $\boldsymbol{q}_{1000} = (\lambda_2 + \lambda_3)/2$, $\boldsymbol{q}_{0100} = (\lambda_1 + \lambda_3)/2$, $\boldsymbol{q}_{0010} = (\lambda_1 + \lambda_2)/2$, $\boldsymbol{q}_{1001} = \boldsymbol{q}_{0101} = \boldsymbol{q}_{0011} = \boldsymbol{q}_{0001} = \boldsymbol{q}_{0000} = 0$, is a such solution.

(iii). One hand, if the equation (35) have some solutions $\boldsymbol{q}$ such that $SP_{\boldsymbol{q}} \subset E_4$, then from

$$\sum_{j=1}^{4} \delta_j \geq 0, \delta_4 + \min_{1\leq j\leq 3}\{\delta_j\} \geq 0,$$

for any $\alpha \in E_4$, we know that $\sum_{j=1}^{4} \lambda_j \geq 0$, $\lambda_4 + \min_{1\leq j\leq 3}\{\lambda_j\} \geq 0$.

On the other hand, we assume $\sum_{j=1}^{4} \lambda_j \geq 0$, $\lambda_4 + \min_{1\leq j\leq 3}\{\lambda_j\} \geq 0$. Then, $\boldsymbol{q}_{0110} = (\lambda_1 + \lambda_4)/2$, $\boldsymbol{q}_{1010} = (\lambda_2 + \lambda_4)/2$, $\boldsymbol{q}_{0011} = (\lambda_3 + \lambda_4)/2$, $\boldsymbol{q}_{0001} = \sum_{j=1}^{4} \lambda_j/2$, $\boldsymbol{q}_{0000} = \boldsymbol{q}_{1000} = \boldsymbol{q}_{0100} = \boldsymbol{q}_{0010} = 0$, is a solution. △△

Let $\mho(\delta_1, \ldots, \delta_k)$ denote the number of the odd integers in $\delta_1, \ldots, \delta_k$, and write

$$\mu(\delta_1, \delta_2, \delta_3, \delta_4) = \begin{cases} 1, & \text{if } \mho(\delta_1, \delta_2, \delta_3, \delta_4) = 2, \\ 0, & \text{otherwise,} \end{cases} \tag{39}$$

$$\nu(\delta_1, \delta_2, \delta_3) = \begin{cases} 0, & \text{if } \mho(\delta_1, \delta_2, \delta_3) = 0, \text{ or } 3 \\ 1, & \text{otherwise,} \end{cases} \tag{40}$$

Theorem 2 *Let $\delta_1 \geq \delta_2 \geq \delta_3 \geq \delta_4$, $\delta_1 > 0$, $\boldsymbol{q} \in Q[\tau]_{\min}$. Then $SP_{\boldsymbol{q}}$ must be a subset of some element of $\{C_1, E_1, E_2, E_3, E_4, D_1, D_2, D_3, D_4\}$. Furthermore,*

(i) if $\delta_2 + \delta_3 + 1 < 0$, and $\delta_1 + \delta_2 + \delta_3 + \delta_4 + 1 + \mu(\delta_1, \delta_2, \delta_3, \delta_4) < 0$, then $SP_{\boldsymbol{q}} \in C_1$ must hold,

(ii) if $\delta_2 + \delta_3 + 1 < 0$, and $\delta_1 + \delta_2 + \delta_3 + \delta_4 + 1 + \mu(\delta_1, \delta_2, \delta_3, \delta_4) \geq 0$, then $SP_{\boldsymbol{q}} \in C_1$ or $\in E_1$,

(iii) if $\delta_2 + \delta_3 + 1 \geq 0$, and $\delta_1 + \delta_4 + 1 < 0$, then $SP_{\boldsymbol{q}} \in C_1$ or $\in D_4$,

(iv) if $\delta_2 + \delta_3 + 1 \geq 0$, $\delta_1 + \delta_4 + 1 \geq 0$, and $\delta_2 + \delta_4 + 1 < 0$, then $SP_{\boldsymbol{q}}$ must be a subset of some element of $\{C_1, E_1, D_4\}$,

(v) if $\delta_2 + \delta_4 + 1 \geq 0$, and $\delta_3 + \delta_4 + 1 < 0$, then $SP_{\boldsymbol{q}}$ must be a subset of some element of $\{C_1, E_1, E_2, D_3, D_4\}$,

(vi) if $\delta_3 + \delta_4 + 1 \geq 0$, and $\delta_1 + \mu(\delta_1, \delta_2, \delta_3, \delta_4) > \delta_2 + \delta_3 + \delta_4 + \nu(\delta_2, \delta_3, \delta_4)$, then $SP_{\boldsymbol{q}}$ must be a subset of some element of $\{C_1, E_1, E_2, E_3, E_4, D_2, D_3, D_4\}$,

Proof: Let $\lambda = \sum_{\alpha\in B^4} \boldsymbol{q}_\alpha (-1)^\alpha$, $\gamma = \Delta(\boldsymbol{q}) = \lambda - \delta$.

From Corollary 1, Lemma 7, Lemma 8 and Theorem 1, we know that $SP_{\boldsymbol{q}}$ must be a subset of some element of $\{C_i, D_i, E_i : i = 1, 2, 3, 4\}$. If $SP_{\boldsymbol{q}} \subset C_i$, $i \geq 2$, then from

Lemma 8 we know $\lambda_i = \delta_i$, and from Lemma 9 we know that $\lambda_1 = \lambda_i$. Furthermore, from any $\alpha \in C_i$ satisfies $\alpha_i = 0$, we know that for any $\alpha \in SP_{\boldsymbol{q}}$ must satisfy $\alpha_1 = 0$, i.e. $SP_{\boldsymbol{q}} \subset C_1$.

If $SP_{\boldsymbol{q}} \subset D_1$, then from Lemma 8 and Lemma 9, we know $\gamma_2 + \gamma_3 + \gamma_4 \leq 1$ and $\sum_{j=2}^{4} \lambda_j \geq \max_{1\leq j \leq 4}\{|\lambda_j|\}$. With respect to that $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ have the same odd-even property, we can get $\lambda_2 + \lambda_3 + \lambda_4 = \delta_2 + \delta_3 + \delta_4 + \nu(\delta_2, \delta_3, \delta_4)$. Clearly, $\lambda_3 + \lambda_4 \geq 0$ and $\lambda_1 \leq \lambda_2 + \lambda_3 + \lambda_4$. Furthermore, from $\lambda_3 + \lambda_4 \geq 0$ and $\gamma_3 + \gamma_4 \leq 1$ we know $\delta_3 + \delta_4 + 1 \geq 0$. On the other hand, it is not difficult to show that

$$\lambda_1 = \delta_1 + \mu(\delta_1, \delta_2, \delta_3, \delta_4) + (1 - \nu(\delta_2, \delta_3, \delta_4))\nu(\delta_1, \delta_2, \delta_3) + 2k,$$

where k is a suitable nonnegative integer. So we have

$$\delta_1 + \mu(\delta_1, \delta_2, \delta_3, \delta_4) \leq \delta_2 + \delta_3 + \delta_4 + \nu(\delta_2, \delta_3, \delta_4).$$

If $SP_{\boldsymbol{q}} \subset E_3$, or E_4,or D_2, then from Lemma 8 and Lemma 9, we know $\lambda_3 + \lambda_4 \geq 0$ and $\gamma_3 + \gamma_4 \leq 1$. So that $\delta_3 + \delta_4 + 1 \geq 0$.

If $SP_{\boldsymbol{q}} \subset E_2$,or D_3, then from Lemma 8 and Lemma 9, we know $\lambda_2 + \lambda_4 \geq 0$ and $\gamma_2 + \gamma_4 \leq 1$. So that $\delta_2 + \delta_4 + 1 \geq 0$.

If $SP_{\boldsymbol{q}} \subset D_4$, then from Lemma 8 and Lemma 9, we know $\lambda_2 + \lambda_3 \geq 0$ and $\gamma_2 + \gamma_3 \leq 1$. So that $\delta_2 + \delta_3 + 1 \geq 0$.

If $SP_{\boldsymbol{q}} \subset E_1$, then from Lemma 8 and Lemma 9, we know $\lambda_1 + \lambda_4 \geq 0$, $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 \geq 0$ and $\gamma \in E_1$. So that $\delta_1 + \delta_4 + 1 \geq 0$ and $\delta_1 + \delta_2 + \delta_3 + \delta_4 + 1 + \mu(\delta_1, \delta_2, \delta_3, \delta_4) \geq 0$.

According the above argument, we can show this theorem easily. △△

Theorem 3 *Let λ be a 4-tuple over integers, its integrants have same odd-even property. For the equation*

$$\sum_{\alpha \in B^4} \boldsymbol{q}_\alpha (-1)^\alpha = \lambda, \tag{41}$$

1. if $\lambda_1 \geq \max_{2\leq j\leq 4}\{|\lambda_j|\}$, then all the nonnegative integral solutions $\boldsymbol{q}$ with $SP_{\boldsymbol{q}} \in C_1$ are

$$\begin{cases} \boldsymbol{q}_{0110} = r, \ \boldsymbol{q}_{0111} = k - r - (\lambda_2 + \lambda_3)/2, \\ \boldsymbol{q}_{0100} = s, \ \boldsymbol{q}_{0101} = (\lambda_1 + \lambda_3)/2 - k - s, \\ \boldsymbol{q}_{0010} = t, \ \boldsymbol{q}_{0011} = (\lambda_1 + \lambda_2)/2 - k - t, \\ \boldsymbol{q}_{0000} = (\lambda_1 + \lambda_4)/2 - r - s - t, \ \boldsymbol{q}_{0001} = k - \boldsymbol{q}_{0000}, \\ \boldsymbol{q}_{1001} = \boldsymbol{q}_{1000} = \boldsymbol{q}_{1011} = \boldsymbol{q}_{1010} = \boldsymbol{q}_{1101} = \boldsymbol{q}_{1100} = \boldsymbol{q}_{1111} = \boldsymbol{q}_{1110} = 0. \end{cases} \tag{42}$$

where r, s, t are nonnegative integers satisfying

$$(\lambda_1 + \lambda_4)/2 - k \leq r + s + t \leq (\lambda_1 + \lambda_4)/2, \tag{43}$$

$$r \leq k - (\lambda_2 + \lambda_3)/2, \;\; s \leq (\lambda_1 + \lambda_3)/2 - k, \;\; t \leq (\lambda_1 + \lambda_2)/2 - k, \tag{44}$$

and k is a nonnegative integer satisfying

$$(\lambda_2 + \lambda_3)/2 \leq k \leq \min\{(\lambda_1 + \lambda_2)/2, \;(\lambda_1 + \lambda_3)/2\}. \tag{45}$$

Moreover, these solutions belong to $Q[\tau]_{\min}$ if and only if $\gamma = \lambda - \delta \geq 0$, $\gamma_1 = 0$.

2. if $\sum_{j=1}^{3} \lambda_j \geq \max_{1\leq j\leq 4}\{|\lambda_j|\}$, then all the nonnegative integral solutions $\boldsymbol{q}$ with $SP\boldsymbol{q} \in D_4$ are

$$\begin{cases} \boldsymbol{q}_{1000} = r, \;\; \boldsymbol{q}_{1001} = (\lambda_2 + \lambda_3)/2 - k - r, \\ \boldsymbol{q}_{0100} = s, \;\; \boldsymbol{q}_{0101} = (\lambda_1 + \lambda_3)/2 - k - s, \\ \boldsymbol{q}_{0010} = t, \;\; \boldsymbol{q}_{0011} = (\lambda_1 + \lambda_2)/2 - k - t, \\ \boldsymbol{q}_{0000} = \sum_{j=1}^{4} \lambda_j/2 - k - r - s - t, \;\; \boldsymbol{q}_{0001} = k - \boldsymbol{q}_{0000}, \\ \boldsymbol{q}_{0111} = \boldsymbol{q}_{0110} = \boldsymbol{q}_{1011} = \boldsymbol{q}_{1010} = \boldsymbol{q}_{1101} = \boldsymbol{q}_{1100} = \boldsymbol{q}_{1111} = \boldsymbol{q}_{1110} = 0. \end{cases} \tag{46}$$

where r, s, t are nonnegative integers satisfying

$$\sum_{j=1}^{4} \lambda_j/2 - 2k \leq r + s + t \leq \sum_{j=1}^{4} \lambda_j/2 - k, \tag{47}$$

$$r \leq (\lambda_2 + \lambda_3)/2 - k, \;\; s \leq (\lambda_1 + \lambda_3)/2 - k, \;\; t \leq (\lambda_1 + \lambda_2)/2 - k, \tag{48}$$

and k is a nonnegative integer satisfying

$$2k \leq \sum_{j=1}^{3} \lambda_j - \max_{1\leq j\leq 4}\{|\lambda_j|\} \tag{49}$$

Moreover, these solutions belong to $Q[\tau]_{\min}$ if and only if $\gamma = \lambda - \delta \geq 0$, and $\gamma_1 + \gamma_2 + \gamma_3 \leq 1$.

3. if $\sum_{j=1}^{4}\lambda_j \geq 0$, *and* $\lambda_1+\min_{2\leq j\leq 4}\{\lambda_j\} \geq 0$, *then all the nonnegative integral solutions* $\boldsymbol{q}$ *with* $SP_{\boldsymbol{q}} \in E_1$ *are*

$$\begin{cases} \boldsymbol{q}_{0001} = \begin{cases} r+(\lambda_2+\lambda_3)/2-k & \textit{if } (\lambda_2+\lambda_3)/2 \geq k, \\ r & \textit{if } (\lambda_2+\lambda_3)/2 \leq k, \end{cases} \\ \boldsymbol{q}_{0110} = \boldsymbol{q}_{0001}-(\lambda_2+\lambda_3)/2+k, \\ \boldsymbol{q}_{0100} = \begin{cases} s+(\lambda_3+\lambda_4)/2-k & \textit{if } (\lambda_3+\lambda_4)/2 \geq k, \\ s & \textit{if } (\lambda_3+\lambda_4)/2 \leq k, \end{cases} \\ \boldsymbol{q}_{0011} = \boldsymbol{q}_{0100}-(\lambda_3+\lambda_4)/2+k, \\ \boldsymbol{q}_{0010} = \begin{cases} t+(\lambda_2+\lambda_4)/2-k & \textit{if } (\lambda_2+\lambda_4)/2 \geq k, \\ t & \textit{if } (\lambda_2+\lambda_4)/2 \leq k, \end{cases} \\ \boldsymbol{q}_{0101} = \boldsymbol{q}_{0010}-(\lambda_2+\lambda_4)/2+k, \\ \boldsymbol{q}_{0000} = (\lambda_1-\Psi(k)+k)/2-r-s-t, \ \boldsymbol{q}_{1000} = k-\boldsymbol{q}_{0000}, \\ \boldsymbol{q}_{1110} = \boldsymbol{q}_{1001} = \boldsymbol{q}_{1011} = \boldsymbol{q}_{1100} = \boldsymbol{q}_{1101} = \boldsymbol{q}_{1010} = \boldsymbol{q}_{1111} = \boldsymbol{q}_{0111} = 0. \end{cases} \tag{50}$$

where

$$\Psi(k) = |(\lambda_2+\lambda_3)/2-k|+|(\lambda_3+\lambda_4)/2-k|+|(\lambda_2+\lambda_4)/2-k|,$$

r, s, t *are nonnegative integers satisfying*

$$(\lambda_1-\Psi(k)-k)/2 \leq r+s+t \leq (\lambda_1-\Psi(k)+k)/2, \tag{51}$$

and k *is a nonnegative integer satisfying*

$$\max\{\sum_{j=1}^{4}\lambda_j/2, \lambda_2, \lambda_3, \lambda_4\}-\lambda_1 \leq 2k \leq \sum_{j=1}^{4}\lambda_j, \tag{52}$$

Moreover, these solutions belong to $Q[\tau]_{\min}$ *if and only if* $\gamma = \lambda-\delta \geq 0$, *and* $\gamma \in E_1$.

Proof: 1. From lemma 9, we know the equation have some solutions $\boldsymbol{q}$ with $SP_{\boldsymbol{q}} \in C_1$. And by Lemma 8, these solutions belong to $Q[\tau]_{\min}$ if and only if $\gamma = \lambda-\delta \geq 0$, $\gamma_1 = 0$.

Now we assume that $\boldsymbol{q}$ with $SP_{\boldsymbol{q}} \in C_1$ is a solution of (41). Let $\boldsymbol{q}_{0000}+\boldsymbol{q}_{0001} = k$. Then we have

$$\begin{cases} \boldsymbol{q}_{0110}+\boldsymbol{q}_{0111} = k-(\lambda_2+\lambda_3)/2, \\ \boldsymbol{q}_{0100}+\boldsymbol{q}_{0101} = (\lambda_1+\lambda_3)/2-k, \\ \boldsymbol{q}_{0010}+\boldsymbol{q}_{0011} = (\lambda_1+\lambda_2)/2-k, \\ \boldsymbol{q}_{0110}+\boldsymbol{q}_{0100}+\boldsymbol{q}_{0010}+\boldsymbol{q}_{0000} = (\lambda_1+\lambda_4)/2. \end{cases}$$

So k satisfies (45). Futhermore, let $\boldsymbol{q}_{0110} = r$, $\boldsymbol{q}_{0100} = s$, $\boldsymbol{q}_{0010} = t$, then (42), (43) and (44) hold.

On the other hand, if k, r, s, t satisfy (45), (43) and (44), it is not difficult to show that (42) gives a nonnegative integral vector $\boldsymbol{q}$ with $SP_{\boldsymbol{q}} \in C_1$, and $\boldsymbol{q}$ satisfy the equation.

2. From lemma 9, we know the equation have some solutions $\boldsymbol{q}$ with $SP_{\boldsymbol{q}} \in D_4$. And by Lemma 8, these solutions belong to $Q[\tau]_{\min}$ if and only if $\gamma = \lambda - \delta \geq 0$, and $\gamma_1 + \gamma_2 + \gamma_3 \leq 1$.

Now we assume that $\boldsymbol{q}$ with $SP_{\boldsymbol{q}} \in D_4$ is a solution of (41). Let $\boldsymbol{q}_{0000} + \boldsymbol{q}_{0001} = k$. Then we have

$$\begin{cases} \boldsymbol{q}_{1000} + \boldsymbol{q}_{1001} = (\lambda_2 + \lambda_3)/2 - k, \\ \boldsymbol{q}_{0100} + \boldsymbol{q}_{0101} = (\lambda_1 + \lambda_3)/2 - k, \\ \boldsymbol{q}_{0010} + \boldsymbol{q}_{0011} = (\lambda_1 + \lambda_2)/2 - k, \\ \boldsymbol{q}_{1000} + \boldsymbol{q}_{0100} + \boldsymbol{q}_{0010} + \boldsymbol{q}_{0000} = \sum_{j=1}^{4} \lambda_j/2 - k, \\ \boldsymbol{q}_{1001} + \boldsymbol{q}_{0101} + \boldsymbol{q}_{0011} + \boldsymbol{q}_{0001} = \sum_{j=1}^{3} \lambda_j/2 - \lambda_4/2 - k. \end{cases}$$

So k satisfy (49). Let $\boldsymbol{q}_{1000} = r$, $\boldsymbol{q}_{0100} = s$, $\boldsymbol{q}_{0010} = t$, then r, s, t satysfy (47) and (48). Clearly, (46) also holds.

On the other hand, if k, r, s, t satisfy (46), (44) and (45), it is not difficult to show that (43) gives a nonnegative integral vector $\boldsymbol{q}$ with $SP_{\boldsymbol{q}} \in D_4$, and $\boldsymbol{q}$ satisfy the equation.

3. From lemma 9, we know the equation have some solutions $\boldsymbol{q}$ with $SP_{\boldsymbol{q}} \in E_1$. And by Lemma 8, these solutions belong to $Q[\tau]_{\min}$ if and only if $\gamma = \lambda - \delta \geq 0$, and $\gamma \in E_1$.

Now we assume that $\boldsymbol{q}$ with $SP_{\boldsymbol{q}} \in E_1$ is a solution of (41). Let $\boldsymbol{q}_{0000} + \boldsymbol{q}_{1000} = k$. Then we have

$$\begin{cases} \boldsymbol{q}_{0001} - \boldsymbol{q}_{0110} = (\lambda_2 + \lambda_3)/2 - k, \\ \boldsymbol{q}_{0100} - \boldsymbol{q}_{0011} = (\lambda_3 + \lambda_4)/2 - k, \\ \boldsymbol{q}_{0010} - \boldsymbol{q}_{0101} = (\lambda_2 + \lambda_4)/2 - k, \\ \boldsymbol{q}_{0001} + \boldsymbol{q}_{0100} + \boldsymbol{q}_{0010} + \boldsymbol{q}_{0000} = \sum_{j=1}^{4} \lambda_j/2 - k, \\ \boldsymbol{q}_{0110} + \boldsymbol{q}_{0011} + \boldsymbol{q}_{0101} + \boldsymbol{q}_{0000} = \lambda_1/2 - \sum_{j=2}^{4} \lambda_j/2 + 2k, \\ \boldsymbol{q}_{0001} + \boldsymbol{q}_{0011} + \boldsymbol{q}_{0101} + \boldsymbol{q}_{0000} = (\lambda_1 - \lambda_4)/2 + k, \\ \boldsymbol{q}_{0110} + \boldsymbol{q}_{0100} + \boldsymbol{q}_{0101} + \boldsymbol{q}_{0000} = (\lambda_1 - \lambda_2)/2 + k, \\ \boldsymbol{q}_{0110} + \boldsymbol{q}_{0011} + \boldsymbol{q}_{0010} + \boldsymbol{q}_{0000} = (\lambda_1 - \lambda_3)/2 + k. \end{cases}$$

So (52) must hold. Moreover, if we write $\min\{\boldsymbol{q}_{0001}, \boldsymbol{q}_{0110}\} = r$, $\min\{\boldsymbol{q}_{0100}, \boldsymbol{q}_{0011}\} = s$, $\min\{\boldsymbol{q}_{0010}, \boldsymbol{q}_{0101}\} = t$, then we can get

$$\begin{aligned} \lambda_1 &= \boldsymbol{q}_{0001} + \boldsymbol{q}_{0110} + \boldsymbol{q}_{0100} + \boldsymbol{q}_{0011} + \boldsymbol{q}_{0010} + \boldsymbol{q}_{0101} + \boldsymbol{q}_{0000} - \boldsymbol{q}_{1000} \\ &= \Psi(k) - k + 2(r + s + t + \boldsymbol{q}_{0000}). \end{aligned}$$

With respect to $0 \leq \boldsymbol{q}_{0000} \leq k$, we know that r, s, t satisfy (51). Clearly, (50) holds.

On the other hand, we assume that k satisfy (52). For convenient, we suppose that $\lambda_2 \geq \lambda_3 \geq \lambda_4$. Then we have

$$\Psi(k) - k = \begin{cases} \lambda_2 + \lambda_3 + \lambda_4 - 4k & \text{if } k \leq (\lambda_3 + \lambda_4)/2, \\ \lambda_2 - 2k & \text{if } (\lambda_3 + \lambda_4)/2 \leq k \leq (\lambda_2 + \lambda_4)/2, \\ -\lambda_4 & \text{if } (\lambda_2 + \lambda_4)/2 \leq k \leq (\lambda_2 + \lambda_3)/2, \\ 2k - (\lambda_2 + \lambda_3 + \lambda_4) & \text{if } k \geq (\lambda_2 + \lambda_3)/2, \end{cases}$$

so that $\Psi(k) - k \leq \lambda_1$. Furthermore, if r, s, t satisfy (51), then (50) gives a nonnegative integral vector $\boldsymbol{q}$ with $SP_{\boldsymbol{q}} \in E_1$, and $\boldsymbol{q}$ satisfy the equation. △△

According to Theorem 2 and Theorem 3, we have completely determined all the 2^h-tuples $\boldsymbol{q}$ in $Q[\tau]_{\min}$ for the case $h = 4$.

4 An algorithm for computing L_1^τ

Following the the discussions of section 2 and section 3, here we propose an algorithm for computing L_1^τ with associating (9). **Algorithm of Main Program**.

INPUT: The hard-decision sequence $\boldsymbol{z}= (z_1, \ldots, z_N)$, the relative reliability sequence $\boldsymbol{r}= (r_1, \ldots, r_N)$, the reference codewords $\boldsymbol{u}^1$,$\boldsymbol{u}^2$,$\boldsymbol{u}^3$,$\boldsymbol{u}^4$, and the radiuses d_1, d_2, d_3, d_4 and $\tau_1, \tau_2, \tau_3, \tau_4$.
OUTPUT: L_1^τ.
Step 1
1. Re-order the reliability sequence $\boldsymbol{r}= (r_1, \ldots, r_N)$, according to the increasing order $r_i \leq r_j$, for $1 \leq i < j \leq N$.
2. Re-order the bits of the reference codewords $\boldsymbol{u}^1$,$\boldsymbol{u}^2$,$\boldsymbol{u}^3$,$\boldsymbol{u}^4$ according to 1.

Step 2
1. Compute $n(\boldsymbol{u}^1)$,$n(\boldsymbol{u}^2)$,$n(\boldsymbol{u}^3)$,$n(\boldsymbol{u}^4)$.
2. Compute $\delta_1, \delta_2, \delta_3, \delta_4$, according to (28).
3. Re-order the sequence $\delta = (\delta_1, \delta_2, \delta_3, \delta_4)$ such that $\delta_1 \geq \delta_2 \geq \delta_3 \geq \delta_4$.
4. If $\delta_1 \leq 0$, then $Q_{\min} = \{(0, \ldots, 0)\}$ and $L_1^\tau = 0$. Otherwise, re-order the order of the reference codewords $\boldsymbol{u}^1$,$\boldsymbol{u}^2$,$\boldsymbol{u}^3$,$\boldsymbol{u}^4$ according to 3.

Step 3
1. Generate the sets $\overline{D}_\alpha$, for $\alpha \in B^4$, according to (5).
2. Compute n_α, for $\alpha \in B^4$, according to (6).

Step 4
1. According to Theorem 2, go to suitable sub-programs.

2. Compare the minimums generated by the sub-programs, output L_1^{τ}.

$\triangle\triangle$

The following Sub-program C is to compute the minimum of $\underline{L}[\boldsymbol{u}]$ over those minimal $\boldsymbol{q}$ with $SP_{\boldsymbol{q}} \subset C_1$.

Algorithm of Sub-program C.

INPUT: The reliability sequence $\boldsymbol{r} = (r_1, \ldots, r_N)$ generated in the 1 of Step 1 of Main program, the sequence $\delta = (\delta_1, \delta_2, \delta_3, \delta_4)$ generated in the 3 of Step 2 of Main program, the sets $\overline{D}_\alpha$, $\alpha \in B^4$ generated in the 1 of Step 3 of Main program, n_α, $\alpha \in B^4$ generated in the 2 of Step 3 of Main program.
OUTPUT: The minimum of $\underline{L}[\boldsymbol{u}]$ over those minimal $\boldsymbol{q}$ with $SP_{\boldsymbol{q}} \subset C_1$.
Step 1
Determine all the probable γ and λ according to Lemma 8, Lemma 9 and the odd-even property of the components of λ.
Step 2
For every λ generated in Step 1, generate the solutions $\boldsymbol{q}$ according to 1 of Theorem 3.
Step 3

1. For every $\boldsymbol{q}$ generated in Step 2, compute $\underline{L}[\boldsymbol{u}] = \sum_{i \in \cup_{\alpha \in B^h} \overline{D}_\alpha^{(q_\alpha)}} r_i$ if $\boldsymbol{q}$ satisfies (9).
2. Output the minimum of $\underline{L}[\boldsymbol{u}]$'s generated in 1. $\triangle\triangle$

The following Sub-program D is to compute the minimum of $\underline{L}[\boldsymbol{u}]$ over those minimal $\boldsymbol{q}$ with $SP_{\boldsymbol{q}} \subset D_m$, $1 \leq m \leq 4$.

Algorithm of Sub-program D.

INPUT: The reliability sequence $\boldsymbol{r} = (r_1, \ldots, r_N)$ generated in the 1 of Step 1 of Main program, the sequence $\delta = (\delta_1, \delta_2, \delta_3, \delta_4)$ generated in the 3 of Step 2 of Main program, the sets $\overline{D}_\alpha$, $\alpha \in B^4$ generated in the 1 of Step 3 of Main program, n_α, $\alpha \in B^4$ generated in the 2 of Step 3 of Main program, the positive integer m.
OUTPUT: The minimum of $\underline{L}[\boldsymbol{u}]$ over those minimal $\boldsymbol{q}$ with $SP_{\boldsymbol{q}} \subset D_m$.
Step 1
Determine all the probable γ and λ according to Lemma 8, Lemma 9 and the odd-even property of the components of λ.

Step 2

Change the values of λ_4 and λ_m for every λ generated in Step 1.

Step 3

For every λ generated in Step 2, generate the solutions $\boldsymbol{q}$ according to 2 of Theorem 3.

Step 4

Change the m-th bit and the fourth bit in the index α of every component $\boldsymbol{q}_\alpha$ of $\boldsymbol{q}$ generated in Step 3.

Step 5

1. For every $\boldsymbol{q}$ generated in Step 4, compute $\underline{L}[\boldsymbol{u}] = \sum_{i \in \cup_{\alpha \in B^h} \overline{D}_\alpha^{(q_\alpha)}} r_i$ if $\boldsymbol{q}$ satisfies (9).
2. Output the minimum of $\underline{L}[\boldsymbol{u}]$'s generated in 1. △△

The following Sub-program E is to compute the minimum of $\underline{L}[\boldsymbol{u}]$ over those minimal $\boldsymbol{q}$ with $SP_{\boldsymbol{q}} \subset E_m$, $1 \leq m \leq 4$.

Algotithm of Sub-program E.

INPUT: The reliability sequence $\boldsymbol{r}= (r_1, \ldots, r_N)$ generated in the 1 of Step 1 of Main program, the sequence $\delta = (\delta_1, \delta_2, \delta_3, \delta_4)$ generated in the 3 of Step 2 of Main program, the sets $\overline{D}_\alpha$, $\alpha \in B^4$ generated in the 1 of Step 3 of Main program, n_α, $\alpha \in B^4$ generated in the 2 of Step 3 of Main program, the positive integer m.

OUTPUT: The minimum of $\underline{L}[\boldsymbol{u}]$ over those minimal $\boldsymbol{q}$ with $SP_{\boldsymbol{q}} \subset E_m$.

Step 1

Determine all the probable γ and λ according to Lemma 8, Lemma 9 and the odd-even property of the components of λ.

Step 2

Change the values of λ_1 and λ_m for every λ generated in Step 1.

Step 3

For every λ generated in Step 2, generate the solutions $\boldsymbol{q}$ according to 3 of Theorem 3.

Step 4

Change the m-th bit and the first bit in the index α of every component $\boldsymbol{q}_\alpha$ of $\boldsymbol{q}$ generated in Step 3.

Step 5

1. For every $\boldsymbol{q}$ generated in Step 4, compute $\underline{L}[\boldsymbol{u}] = \sum_{i \in \cup_{\alpha \in B^h} \overline{D}_\alpha^{(q_\alpha)}} r_i$ if $\boldsymbol{q}$ satisfies (9).
2. Output the minimum of $\underline{L}[\boldsymbol{u}]$'s generated in 1. △△

5 The computation of L_2^τ

Let

$$\overline{Q}^\tau = \{\boldsymbol{q} \in \overline{Q}_\tau : \boldsymbol{q}_\alpha \cdot \boldsymbol{q}_{\overline{\alpha}} = 0, \text{for } \alpha \in B^h\}, \tag{53}$$

where $\overline{\alpha}$ denotes the sequence in B^h with $\overline{\alpha}_i = 1 - \alpha_i$, $i = 1, \ldots, h$. Clearly, we have

$$\min_{\boldsymbol{q} \in \overline{Q}^\tau_{\min}} \sum_{i \in \cup_{\alpha \in B^h} \overline{D}_\alpha^{(q_\alpha)}} \boldsymbol{r}_i = L_2^\tau. \tag{54}$$

Let $\tau' \in T(\tau)$, we shall consider $Q(\tau') \cap \overline{Q}^\tau$ below. Let $\lambda = \lambda(\tau')$ denote the h-tuple with

$$\lambda_i = \tau'_i + w_{j_0+1} - n(\boldsymbol{u}^i), \quad i = 1, \ldots, h. \tag{55}$$

If the even-odd properties of the components of λ are different, then $Q(\tau')$ and $Q(\tau') \cap \overline{Q}^\tau$ are empty sets. Now we suppose the components of λ have same even-odd property.

If $h \geq 4$, the set of $\overline{Q}^\tau$ will be quite complicated. Here we only consider the cases of $h = 2$ and $h = 3$. If $h = 2$, there are 4 cases as following.

$$\begin{cases} \boldsymbol{q}_{00} + \boldsymbol{q}_{01} = \lambda_1, \\ \boldsymbol{q}_{00} - \boldsymbol{q}_{01} = \lambda_2, \end{cases} \tag{56}$$

$$\begin{cases} \boldsymbol{q}_{00} - \boldsymbol{q}_{10} = \lambda_1, \\ \boldsymbol{q}_{00} + \boldsymbol{q}_{10} = \lambda_2, \end{cases} \tag{57}$$

$$\begin{cases} -\boldsymbol{q}_{11} + \boldsymbol{q}_{01} = \lambda_1, \\ -\boldsymbol{q}_{11} - \boldsymbol{q}_{01} = \lambda_2, \end{cases} \tag{58}$$

$$\begin{cases} -\boldsymbol{q}_{11} - \boldsymbol{q}_{10} = \lambda_1, \\ -\boldsymbol{q}_{11} + \boldsymbol{q}_{10} = \lambda_2, \end{cases} \tag{59}$$

2.1 Equation (55) has nonnegative integral solution if and only if

$$\lambda_1 + \lambda_2 \geq 0, \ \lambda_1 \geq \lambda_2. \tag{60}$$

And the solution is

$$\begin{cases} \boldsymbol{q}_{00} = (\lambda_1 + \lambda_2)/2, \\ \boldsymbol{q}_{01} = (\lambda_1 - \lambda_2)/2. \end{cases} \tag{61}$$

2.2 Equation (56) has nonnegative integral solution if and only if

$$\lambda_1 + \lambda_2 \geq 0,\ \ \lambda_1 \leq \lambda_2. \tag{62}$$

And the solution is

$$\begin{cases} \boldsymbol{q}_{00} = (\lambda_1 + \lambda_2)/2, \\ \boldsymbol{q}_{10} = (\lambda_2 - \lambda_1)/2. \end{cases} \tag{63}$$

2.3 Equation (57) has nonnegative integral solution if and only if

$$\lambda_1 + \lambda_2 \leq 0,\ \ \lambda_1 \geq \lambda_2. \tag{64}$$

And the solution is

$$\begin{cases} \boldsymbol{q}_{11} = -(\lambda_1 + \lambda_2)/2, \\ \boldsymbol{q}_{01} = (\lambda_1 - \lambda_2)/2. \end{cases} \tag{65}$$

2.4 Equation (58) has nonnegative integral solution if and only if

$$\lambda_1 + \lambda_2 \leq 0,\ \ \lambda_1 \leq \lambda_2. \tag{66}$$

And the solution is

$$\begin{cases} \boldsymbol{q}_{00} = -(\lambda_1 + \lambda_2)/2, \\ \boldsymbol{q}_{10} = (\lambda_2 - \lambda_1)/2. \end{cases} \tag{67}$$

If $h = 3$, there are 16 cases as following.

$$\begin{cases} \boldsymbol{q}_{000} + \boldsymbol{q}_{001} + \boldsymbol{q}_{010} - \boldsymbol{q}_{100} = \lambda_1, \\ \boldsymbol{q}_{000} + \boldsymbol{q}_{001} - \boldsymbol{q}_{010} + \boldsymbol{q}_{100} = \lambda_2, \\ \boldsymbol{q}_{000} - \boldsymbol{q}_{001} + \boldsymbol{q}_{010} + \boldsymbol{q}_{100} = \lambda_3, \end{cases} \tag{68}$$

$$\begin{cases} \boldsymbol{q}_{111} + \boldsymbol{q}_{110} + \boldsymbol{q}_{101} - \boldsymbol{q}_{011} = -\lambda_1, \\ \boldsymbol{q}_{111} + \boldsymbol{q}_{110} - \boldsymbol{q}_{101} + \boldsymbol{q}_{011} = -\lambda_2, \\ \boldsymbol{q}_{111} - \boldsymbol{q}_{110} + \boldsymbol{q}_{101} + \boldsymbol{q}_{011} = -\lambda_3, \end{cases} \tag{69}$$

$$\begin{cases} \boldsymbol{q}_{000} + \boldsymbol{q}_{001} + \boldsymbol{q}_{010} + \boldsymbol{q}_{011} = \lambda_1, \\ \boldsymbol{q}_{000} + \boldsymbol{q}_{001} - \boldsymbol{q}_{010} - \boldsymbol{q}_{011} = \lambda_2, \\ \boldsymbol{q}_{000} - \boldsymbol{q}_{001} + \boldsymbol{q}_{010} - \boldsymbol{q}_{011} = \lambda_3, \end{cases} \tag{70}$$

$$\begin{cases} \boldsymbol{q}_{111} + \boldsymbol{q}_{110} + \boldsymbol{q}_{101} + \boldsymbol{q}_{100} = -\lambda_1, \\ \boldsymbol{q}_{111} + \boldsymbol{q}_{110} - \boldsymbol{q}_{101} - \boldsymbol{q}_{100} = -\lambda_2, \\ \boldsymbol{q}_{111} - \boldsymbol{q}_{110} + \boldsymbol{q}_{101} - \boldsymbol{q}_{100} = -\lambda_3, \end{cases} \tag{71}$$

$$\begin{cases} \boldsymbol{q}_{000} + \boldsymbol{q}_{001} - \boldsymbol{q}_{101} - \boldsymbol{q}_{100} = \lambda_1, \\ \boldsymbol{q}_{000} + \boldsymbol{q}_{001} + \boldsymbol{q}_{101} + \boldsymbol{q}_{100} = \lambda_2, \\ \boldsymbol{q}_{000} - \boldsymbol{q}_{001} - \boldsymbol{q}_{101} + \boldsymbol{q}_{100} = \lambda_3, \end{cases} \tag{72}$$

$$\begin{cases} \boldsymbol{q}_{111} + \boldsymbol{q}_{110} - \boldsymbol{q}_{010} - \boldsymbol{q}_{011} = -\lambda_1, \\ \boldsymbol{q}_{111} + \boldsymbol{q}_{110} + \boldsymbol{q}_{010} + \boldsymbol{q}_{011} = -\lambda_2, \\ \boldsymbol{q}_{111} - \boldsymbol{q}_{110} - \boldsymbol{q}_{010} + \boldsymbol{q}_{011} = -\lambda_3, \end{cases} \tag{73}$$

$$\begin{cases} \boldsymbol{q}_{000} - \boldsymbol{q}_{110} + \boldsymbol{q}_{010} - \boldsymbol{q}_{100} = \lambda_1, \\ \boldsymbol{q}_{000} - \boldsymbol{q}_{110} - \boldsymbol{q}_{010} + \boldsymbol{q}_{100} = \lambda_2, \\ \boldsymbol{q}_{000} + \boldsymbol{q}_{110} + \boldsymbol{q}_{010} + \boldsymbol{q}_{100} = \lambda_3, \end{cases} \tag{74}$$

$$\begin{cases} \boldsymbol{q}_{111} - \boldsymbol{q}_{001} + \boldsymbol{q}_{101} - \boldsymbol{q}_{011} = -\lambda_1, \\ \boldsymbol{q}_{111} - \boldsymbol{q}_{001} - \boldsymbol{q}_{101} + \boldsymbol{q}_{011} = -\lambda_2, \\ \boldsymbol{q}_{111} + \boldsymbol{q}_{001} + \boldsymbol{q}_{101} + \boldsymbol{q}_{011} = -\lambda_3, \end{cases} \tag{75}$$

$$\begin{cases} \boldsymbol{q}_{000} + \boldsymbol{q}_{001} - \boldsymbol{q}_{101} + \boldsymbol{q}_{011} = \lambda_1, \\ \boldsymbol{q}_{000} + \boldsymbol{q}_{001} + \boldsymbol{q}_{101} - \boldsymbol{q}_{011} = \lambda_2, \\ \boldsymbol{q}_{000} - \boldsymbol{q}_{001} - \boldsymbol{q}_{101} - \boldsymbol{q}_{011} = \lambda_3, \end{cases} \tag{76}$$

$$\begin{cases} \boldsymbol{q}_{111} + \boldsymbol{q}_{110} - \boldsymbol{q}_{010} + \boldsymbol{q}_{100} = -\lambda_1, \\ \boldsymbol{q}_{111} + \boldsymbol{q}_{110} + \boldsymbol{q}_{010} - \boldsymbol{q}_{100} = -\lambda_2, \\ \boldsymbol{q}_{111} - \boldsymbol{q}_{110} - \boldsymbol{q}_{010} - \boldsymbol{q}_{100} = -\lambda_3, \end{cases} \tag{77}$$

$$\begin{cases} \boldsymbol{q}_{000} - \boldsymbol{q}_{110} + \boldsymbol{q}_{010} + \boldsymbol{q}_{011} = \lambda_1, \\ \boldsymbol{q}_{000} - \boldsymbol{q}_{110} - \boldsymbol{q}_{010} - \boldsymbol{q}_{011} = \lambda_2, \\ \boldsymbol{q}_{000} + \boldsymbol{q}_{110} + \boldsymbol{q}_{010} - \boldsymbol{q}_{011} = \lambda_3, \end{cases} \tag{78}$$

$$\begin{cases} \boldsymbol{q}_{111} - \boldsymbol{q}_{001} + \boldsymbol{q}_{101} + \boldsymbol{q}_{100} = -\lambda_1, \\ \boldsymbol{q}_{111} - \boldsymbol{q}_{001} - \boldsymbol{q}_{101} - \boldsymbol{q}_{100} = -\lambda_2, \\ \boldsymbol{q}_{111} + \boldsymbol{q}_{001} + \boldsymbol{q}_{101} - \boldsymbol{q}_{100} = -\lambda_3, \end{cases} \tag{79}$$

$$\begin{cases} \boldsymbol{q}_{000} - \boldsymbol{q}_{110} - \boldsymbol{q}_{101} - \boldsymbol{q}_{100} = \lambda_1, \\ \boldsymbol{q}_{000} - \boldsymbol{q}_{110} + \boldsymbol{q}_{101} + \boldsymbol{q}_{100} = \lambda_2, \\ \boldsymbol{q}_{000} + \boldsymbol{q}_{110} - \boldsymbol{q}_{101} + \boldsymbol{q}_{100} = \lambda_3, \end{cases} \tag{80}$$

$$\begin{cases} \boldsymbol{q}_{111} - \boldsymbol{q}_{001} - \boldsymbol{q}_{010} - \boldsymbol{q}_{011} = -\lambda_1, \\ \boldsymbol{q}_{111} - \boldsymbol{q}_{001} + \boldsymbol{q}_{010} + \boldsymbol{q}_{011} = -\lambda_2, \\ \boldsymbol{q}_{111} + \boldsymbol{q}_{001} - \boldsymbol{q}_{010} + \boldsymbol{q}_{011} = -\lambda_3, \end{cases} \tag{81}$$

$$\begin{cases} \boldsymbol{q}_{000} - \boldsymbol{q}_{110} - \boldsymbol{q}_{101} + \boldsymbol{q}_{011} = \lambda_1, \\ \boldsymbol{q}_{000} - \boldsymbol{q}_{110} + \boldsymbol{q}_{101} - \boldsymbol{q}_{011} = \lambda_2, \\ \boldsymbol{q}_{000} + \boldsymbol{q}_{110} - \boldsymbol{q}_{101} - \boldsymbol{q}_{011} = \lambda_3, \end{cases} \tag{82}$$

$$\begin{cases} \boldsymbol{q}_{111} - \boldsymbol{q}_{001} - \boldsymbol{q}_{010} + \boldsymbol{q}_{100} = -\lambda_1, \\ \boldsymbol{q}_{111} - \boldsymbol{q}_{001} + \boldsymbol{q}_{010} - \boldsymbol{q}_{100} = -\lambda_2, \\ \boldsymbol{q}_{111} + \boldsymbol{q}_{001} - \boldsymbol{q}_{010} - \boldsymbol{q}_{100} = -\lambda_3, \end{cases} \tag{83}$$

3.1 Equation (68) has some nonnegative integral solutions if and only if

$$\lambda_1 + \lambda_2 + \lambda_3 \geq \max\{\lambda_1, \lambda_2, \lambda_3\}, \tag{84}$$

and the solutions are

$$\boldsymbol{q}_{001} = (\lambda_1 + \lambda_2)/2 - k,\ \boldsymbol{q}_{010} = (\lambda_1 + \lambda_3)/2 - k,\ \boldsymbol{q}_{100} = (\lambda_2 + \lambda_3)/2 - k, \tag{85}$$

$$\boldsymbol{q}_{000} = k,\ 0 \leq 2k \leq \lambda_1 + \lambda_2 + \lambda_3 - \max\{\lambda_1, \lambda_2, \lambda_3\}. \tag{86}$$

3.2 Equation (69) has some nonnegative integral solutions if and only if

$$\lambda_1 + \lambda_2 + \lambda_3 \leq \min\{\lambda_1, \lambda_2, \lambda_3\}, \tag{87}$$

and the solutions are

$$\boldsymbol{q}_{110} = -(\lambda_1 + \lambda_2)/2 - k,\ \boldsymbol{q}_{101} = -(\lambda_1 + \lambda_3)/2 - k,\ \boldsymbol{q}_{011} = -(\lambda_2 + \lambda_3)/2 - k, \tag{88}$$

$$\boldsymbol{q}_{111} = k,\ 0 \leq 2k \leq \min\{\lambda_1, \lambda_2, \lambda_3\} - (\lambda_1 + \lambda_2 + \lambda_3). \tag{89}$$

3.3 Equation (70) has some nonnegative integral solutions if and only if

$$\lambda_1 \geq \max\{|\lambda_2|, |\lambda_3|\}, \tag{90}$$

and the solutions are

$$\boldsymbol{q}_{001} = (\lambda_1 + \lambda_2)/2 - k,\ \boldsymbol{q}_{010} = (\lambda_1 + \lambda_3)/2 - k,\ \boldsymbol{q}_{011} = k - (\lambda_2 + \lambda_3)/2, \tag{91}$$

$$\boldsymbol{q}_{000} = k,\ \max\{0, \lambda_2 + \lambda_3\} \leq 2k \leq \lambda_1 + \min\{\lambda_2, \lambda_3\}. \tag{92}$$

3.4 Equation (71) has some nonnegative integral solutions if and only if

$$\lambda_1 + \max\{|\lambda_2|, |\lambda_3|\} \leq 0, \tag{93}$$

and the solutions are

$$\boldsymbol{q}_{110} = -(\lambda_1 + \lambda_2)/2 - k,\ \boldsymbol{q}_{101} = -(\lambda_1 + \lambda_3)/2 - k,\ \boldsymbol{q}_{100} = k + (\lambda_2 + \lambda_3)/2, \tag{94}$$

$$\boldsymbol{q}_{111} = k,\ \max\{0, -\lambda_2 - \lambda_3\} \leq 2k \leq -\lambda_1 - \max\{\lambda_2, \lambda_3\}. \tag{95}$$

3.5 Equation (72) has some nonnegative integral solutions if and only if

$$\lambda_2 \geq \max\{|\lambda_1|, |\lambda_3|\}, \tag{96}$$

and the solutions are

$$\boldsymbol{q}_{001} = (\lambda_1+\lambda_2)/2 - k,\ \boldsymbol{q}_{101} = k - (\lambda_1+\lambda_3)/2,\ \boldsymbol{q}_{100} = (\lambda_2+\lambda_3)/2 - k, \tag{97}$$
$$\boldsymbol{q}_{000} = k,\ \max\{0, \lambda_1+\lambda_3\} \le 2k \le \lambda_2 + \min\{\lambda_1, \lambda_3\}. \tag{98}$$

3.6 Equation (73) has some nonnegative integral solutions if and only if

$$\lambda_2 + \max\{|\lambda_1|, |\lambda_3|\} \le 0, \tag{99}$$

and the solutions are

$$\boldsymbol{q}_{110} = -(\lambda_1+\lambda_2)/2 - k,\ \boldsymbol{q}_{010} = k + (\lambda_1+\lambda_3)/2,\ \boldsymbol{q}_{011} = -(\lambda_2+\lambda_3)/2 - k, \tag{100}$$
$$\boldsymbol{q}_{111} = k,\ \max\{0, -\lambda_1-\lambda_3\} \le 2k \le -\lambda_2 - \max\{\lambda_1, \lambda_3\}. \tag{101}$$

3.7 Equation (74) has some nonnegative integral solutions if and only if

$$\lambda_3 \ge \max\{|\lambda_1|, |\lambda_2|\}, \tag{102}$$

and the solutions are

$$\boldsymbol{q}_{110} = k - (\lambda_1+\lambda_2)/2,\ \boldsymbol{q}_{010} = (\lambda_1+\lambda_3)/2 - k,\ \boldsymbol{q}_{100} = (\lambda_2+\lambda_3)/2 - k, \tag{103}$$
$$\boldsymbol{q}_{000} = k,\ \max\{0, \lambda_1+\lambda_2\} \le 2k \le \lambda_3 + \min\{\lambda_1, \lambda_2\}. \tag{104}$$

3.8 Equation (75) has some nonnegative integral solutions if and only if

$$\lambda_3 + \max\{|\lambda_1|, |\lambda_2|\} \le 0, \tag{105}$$

and the solutions are

$$\boldsymbol{q}_{001} = k + (\lambda_1+\lambda_2)/2,\ \boldsymbol{q}_{101} = -(\lambda_1+\lambda_3)/2 - k,\ \boldsymbol{q}_{011} = -(\lambda_2+\lambda_3)/2 - k, \tag{106}$$
$$\boldsymbol{q}_{111} = k,\ \max\{0, -\lambda_1-\lambda_2\} \le 2k \le -\lambda_3 - \max\{\lambda_1, \lambda_2\}. \tag{107}$$

3.9 Equation (76) has some nonnegative integral solutions if and only if

$$\lambda_1+\lambda_2 \ge 0,\ \lambda_3 \le \min\{\lambda_1, \lambda_2\}, \tag{108}$$

and the solutions are

$$\boldsymbol{q}_{001} = (\lambda_1+\lambda_2)/2 - k,\ \boldsymbol{q}_{101} = k - (\lambda_1+\lambda_3)/2,\ \boldsymbol{q}_{011} = k - (\lambda_2+\lambda_3)/2, \tag{109}$$
$$\boldsymbol{q}_{000} = k,\ \lambda_3 + \max\{\lambda_1, \lambda_2, -\lambda_3\} \le 2k \le \lambda_1+\lambda_2. \tag{110}$$

3.10 Equation (77) has some nonnegative integral solutions if and only if

$$\lambda_1+\lambda_2 \le 0,\ \lambda_3 \ge \max\{\lambda_1, \lambda_2\}, \tag{111}$$

and the solutions are

$$\boldsymbol{q}_{110} = -(\lambda_1+\lambda_2)/2 - k,\ \ \boldsymbol{q}_{010} = k + (\lambda_1+\lambda_3)/2,\ \ \boldsymbol{q}_{100} = k + (\lambda_2+\lambda_3)/2, \tag{112}$$

$$\boldsymbol{q}_{111} = k,\ \ -\lambda_3 - \min\{\lambda_1, \lambda_2, -\lambda_3\} \le 2k \le -\lambda_1 - \lambda_2. \tag{113}$$

3.11 Equation (78) has some nonnegative integral solutions if and only if

$$\lambda_1+\lambda_3 \ge 0,\ \ \lambda_2 \le \min\{\lambda_1, \lambda_3\}, \tag{114}$$

and the solutions are

$$\boldsymbol{q}_{110} = k - (\lambda_1+\lambda_2)/2,\ \ \boldsymbol{q}_{010} = (\lambda_1+\lambda_3)/2 - k,\ \ \boldsymbol{q}_{011} = k - (\lambda_2+\lambda_3)/2, \tag{115}$$

$$\boldsymbol{q}_{000} = k,\ \ \lambda_2 + \max\{\lambda_1, -\lambda_2, \lambda_3\} \le 2k \le \lambda_1 + \lambda_3. \tag{116}$$

3.12 Equation (79) has some nonnegative integral solutions if and only if

$$\lambda_1+\lambda_2 \le 0,\ \ \lambda_3 \ge \max\{\lambda_1, \lambda_2\}, \tag{117}$$

and the solutions are

$$\boldsymbol{q}_{001} = k + (\lambda_1+\lambda_2)/2,\ \ \boldsymbol{q}_{101} = -(\lambda_1+\lambda_3)/2 - k,\ \ \boldsymbol{q}_{100} = k + (\lambda_2+\lambda_3)/2, \tag{118}$$

$$\boldsymbol{q}_{111} = k,\ \ -\lambda_2 - \min\{\lambda_1, -\lambda_2, \lambda_3\} \le 2k \le -\lambda_1 - \lambda_3. \tag{119}$$

3.13 Equation (80) has some nonnegative integral solutions if and only if

$$\lambda_2+\lambda_3 \ge 0,\ \ \lambda_1 \le \min\{\lambda_2, \lambda_3\}, \tag{120}$$

and the solutions are

$$\boldsymbol{q}_{110} = k - (\lambda_1+\lambda_2)/2,\ \ \boldsymbol{q}_{101} = k - (\lambda_1+\lambda_3)/2,\ \ \boldsymbol{q}_{100} = (\lambda_2+\lambda_3)/2 - k, \tag{121}$$

$$\boldsymbol{q}_{000} = k,\ \ \lambda_1 + \max\{-\lambda_1, \lambda_2, \lambda_3\} \le 2k \le \lambda_2 + \lambda_3. \tag{122}$$

3.14 Equation (81) has some nonnegative integral solutions if and only if

$$\lambda_2+\lambda_3 \le 0,\ \ \lambda_1 \ge \max\{\lambda_2, \lambda_3\}, \tag{123}$$

and the solutions are

$$\boldsymbol{q}_{001} = k + (\lambda_1+\lambda_2)/2,\ \ \boldsymbol{q}_{010} = -(\lambda_1+\lambda_3)/2 - k,\ \ \boldsymbol{q}_{011} = k + (\lambda_2+\lambda_3)/2, \tag{124}$$

$$\boldsymbol{q}_{111} = k,\ \ -\lambda_1 - \min\{-\lambda_1, \lambda_2, \lambda_3\} \le 2k \le -\lambda_2 - \lambda_3. \tag{125}$$

3.15 Equation (82) always has nonnegative integral solutions, but only if

$$\lambda_1+\lambda_2 < 0,\ \ \lambda_2+\lambda_3 < 0,\ \ \lambda_1+\lambda_3 < 0, \tag{126}$$

equation (82) has nonnegative integral solution $\boldsymbol{q}$ which belongs to $\overline{Q}^{\tau}$ and there is no other $\boldsymbol{q}' \in \overline{Q}(\tau')$ such that $\boldsymbol{q}' \leq \boldsymbol{q}$. And the solution is

$$\boldsymbol{q}_{110} = -(\lambda_1 + \lambda_2)/2,\ \boldsymbol{q}_{101} = -(\lambda_1 + \lambda_3)/2,\ \boldsymbol{q}_{011} = -(\lambda_2 + \lambda_3)/2,\ \boldsymbol{q}_{000} = 0, \qquad (127)$$

3.16 Equation (83) always has nonnegative integral solutions, but only if

$$\lambda_1 + \lambda_2 > 0,\ \lambda_2 + \lambda_3 > 0,\ \lambda_1 + \lambda_3 > 0, \qquad (128)$$

equation (83) has nonnegative integral solution $\boldsymbol{q}$ which belongs to $\overline{Q}^{\tau}$ and there is no other $\boldsymbol{q}' \in \overline{Q}(\tau')$ such that $\boldsymbol{q}' \leq \boldsymbol{q}$. And the solution is

$$\boldsymbol{q}_{001} = (\lambda_1 + \lambda_2)/2,\ \boldsymbol{q}_{010} = (\lambda_1 + \lambda_3)/2,\ \boldsymbol{q}_{100} = (\lambda_2 + \lambda_3)/2,\ \boldsymbol{q}_{111} = 0, \qquad (129)$$

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